Arithmetic intersections on modular curves

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To my beloved wife
and my little angel

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Abstract

An important invariant of modular curves is the arithmetic self-intersection of the relative dualizing sheaf. On the minimal regular model of $X(N)$ this self-intersection is completely described by the usual intersection of some distinguished vertical divisors (geometric contribution) and the evaluation of the canonical Green's function at certain cusps (analytic contribution). The aim of this thesis is to determine each of these contributions in terms of the level N and study the asymptotic behaviour of the self-intersection as N tends to infinity.

Keywords:
Arakelov theory, moduli problems, modular curves, automorphic forms, canonical Green's function, zeta functions.

Zusammenfassung

Eine wichtige Invariante von Modulkurven ist die arithmetische Selbstschnittzahl der relativ dualisierenden Garbe. Auf dem minimalen regulären Modell von $X(N)$ ist diese Selbstschnittzahl durch den gewöhnlichen Schnitt einiger ausgezeichneter vertikaler Divisoren (dem geometrischen Beitrag) und durch die Auswertung der kanonischen Greenschen Funktion an einigen Spitzen (dem analytischen Beitrag) vollständig festgelegt. Das Ziel dieser Arbeit ist es, jeden dieser Beiträge in Abhängigkeit von der Stufe N zu bestimmen und das asymptotische Verhalten der Selbstschnittzahl zu studieren, wenn die Stufe N gegen unendlich geht.

Schlagwörter:
Arakelovtheorie, Modulkurven, automorphe Formen, kanonische Greensche Funktion, Zetafunktionen.

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Introduction

A *modular curve* is a smooth projective algebraic curve X/K defined over the field $K = \mathbb{Q}(\zeta_N)^{\det(H)}$, where N is a positive integer, ζ_N is an N-th root of unity and $H \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is a subgroup. The curve $X_{\mathbb{C}}$, which is obtained from the modular curve X/K after a base change to $\mathbb{C}$, can be regarded as a Riemann surface of the form $\overline{\Gamma\backslash\mathbb{H}}$, namely, the compactification of the space of orbits $\Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup acting on $\mathbb{H}$ by linear fractional transformations.

The classical modular curves $X_0(N)/\mathbb{Q}$, $X_1(N)/\mathbb{Q}$ and $X(N)/\mathbb{Q}(\zeta_N)$ that one encounters in the literature correspond to $H = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$, $\{\pm\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$, and $\{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ respectively. Furthermore these choices of H produce the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ respectively.

Let X/K be a modular curve of genus g and set $S := \mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K. If $g > 0$, then there exists an arithmetic surface $\mathcal{X}/S$ whose generic fiber is isomorphic to X/K. On the arithmetic surface $\mathcal{X}/S$ we have the relative dualizing sheaf which is the invertible $\mathcal{O}_{\mathcal{X}}$-module given by

$$\omega_{\mathcal{X}/S} := \det(\mathcal{C}_{\mathcal{X}/\mathbb{P}^m_S})^\vee \otimes_{\mathcal{O}_{\mathcal{X}}} \det(i^*(\Omega^1_{\mathbb{P}^m_S/S})),$$

where $i : \mathcal{X} \longrightarrow \mathbb{P}^m_S$ is a closed immersion, $\mathcal{C}_{\mathcal{X}/\mathbb{P}^m_S}$ is the conormal sheaf and $\Omega^1_{\mathbb{P}^m_S/S}$ is the sheaf of Kähler differentials. The arithmetic self-intersection of the relative dualizing sheaf $\overline{\omega}^2_{\mathcal{X}/\mathcal{O}_K}$, in the sense of Arakelov theory, defines an important invariant of the modular curve X/K.

Towards the end of the 90's, A. Abbes and E. Ullmo found in [AU97] a formula for this invariant and fully described the case of the modular curve $X_0(N)/\mathbb{Q}$, where N is an odd square-free positive integer relatively prime to 6. More remarkable is the fact that, as the level N tends to infinity, the following asymptotic expansion holds

$$\overline{\omega}^2_{\mathcal{X}_0(N)/\mathbb{Z}} = 3g\log(N) + o(g\log(N)),$$

where $\mathcal{X}_0(N)/\mathbb{Z}$ stands for the arithmetic surface associated to the modular curve $X_0(N)/\mathbb{Q}$.

Based on the ideas of Abbes–Ullmo, H. Mayer worked the case of the modular curve $X_1(N)/\mathbb{Q}(\zeta_N)$ in [May14], where $N = N'qr$ is a positive integer such that $q, r \geq 4$ are relatively prime integers. He showed that, as the level N tends to infinity, the following asymptotic expansion holds

$$\frac{1}{\varphi(N)}\overline{\omega}^2_{\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]} = 3g\log(N) + o(g\log(N)),$$

where $\varphi(N)$ denotes the Euler totient function and $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$ stands for the arithmetic surface associated to the modular curve $X_1(N)/\mathbb{Q}(\zeta_N)$.

Motivated by these remarks and the fact that congruence subgroups are closely related to $\Gamma(N)$, we compute in this thesis the arithmetic self-intersection of the relative dualizing sheaf of the modular curve $X(N)/\mathbb{Q}(\zeta_N)$. We adopt the following guiding philosophy: Invariants of congruence subgroups can be completely determined using data coming from $\Gamma(N)$.

Notations

Let $N \geq 3$ be an integer. The principal congruence subgroup $\Gamma(N)$ is defined by

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1 \bmod N,\ b \equiv c \equiv 0 \bmod N \right\}.$$

Let $X(N)/\mathbb{Q}(\zeta_N)$ be the modular curve associated to $\Gamma(N)$ and denote by $\mathcal{X}(N)/\mathbb{Z}[\zeta_N]$ the corresponding arithmetic surface. Let us write $X(N)_{\mathbb{C}}$ for the compact Riemann surface $\overline{\Gamma(N)\backslash\mathbb{H}}$ of genus g.

We fix an orthonormal basis $\{\alpha_1, \ldots, \alpha_g\}$ of the $\mathbb{C}$-vector space of global holomorphic 1-forms $\Omega^1_{X(N)_{\mathbb{C}}}$. Then the canonical volume form on $X(N)_{\mathbb{C}}$ is given by

$$\mu_{\mathrm{can}} := \frac{i}{2g}\sum_{j=1}^{g} \alpha_j \wedge \overline{\alpha}_j.$$

This definition does not depend on the choice of the basis. By the Kodaira–Spencer isomorphism, we can write μ_{can} in the local coordinate z, as follows

$$\mu_{\mathrm{can}}(z) = F_{\Gamma(N)}(z)\,\mu_{\mathrm{hyp}}(z)$$

with

$$F_{\Gamma(N)}(z) := \frac{\mathrm{Im}(z)^2}{g}\sum_{j=1}^{g} |f_j(z)|^2,$$

where $\{f_1, \ldots, f_g\}$ is an orthonormal basis of the space of cusp forms $\mathcal{S}_2(\Gamma(N))$ of weight 2 with respect to $\Gamma(N)$. The function $F_{\Gamma(N)}(z)$ is called the Arakelov metric on $X(N)_{\mathbb{C}}$.

Let $g_{\mathrm{can}}^{\Gamma(N)}$ be the Green's function associated to the volume form μ_{can}. It is a real valued smooth function on $(X(N)_{\mathbb{C}} \times X(N)_{\mathbb{C}}) \setminus \Delta_{X(N)_{\mathbb{C}}}$ which is uniquely determined by the following conditions:

(i) $\mathrm{dd}^{\mathrm{c}} g_{\mathrm{can}}^{\Gamma(N)}(z,w) + \delta_w(z) = \mu_{\mathrm{can}}(z)$, where $\delta_w(\cdot)$ denotes the Dirac delta distribution;

(ii) $\displaystyle\int\limits_{X(N)_{\mathbb{C}}} g_{\mathrm{can}}^{\Gamma(N)}(z,w)\mu_{\mathrm{can}}(z) = 0.$

Similarly, we obtain a Green's function $g_{\mathrm{can}}^{\Gamma}$ for arbitrary congruence subgroups Γ using the previous constructions mutatis mutandis.

Denote by $(\mathrm{Ell}/\mathbb{Z}[\zeta_N])$ the category of elliptic curves over $\mathrm{Spec}(\mathbb{Z}[\zeta_N])$ and let $[\Gamma(N)]^{\mathrm{can}}$ be the moduli problem of canonical $\Gamma(N)$-structures on the objects of $(\mathrm{Ell}/\mathbb{Z}[\zeta_N])$.

Main results

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup associated to the modular curve X_Γ/K with corresponding subgroup $H \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Denote by $X(\Gamma)$ the Riemann surface $\overline{\Gamma\backslash\mathbb{H}}$ of genus g_Γ. Consider the map $f : X(\Gamma) \longrightarrow \mathbb{P}^1(\mathbb{C})$ given by the j-invariant and let $f^{-1}(1728) = \{P_1, \ldots, P_k\}$ and $f^{-1}(0) = \{Q_1, \ldots, Q_l\}$. In addition, suppose that Γ satisfies the following conditions:

(i) The ramification index of f at the points P_j is constant and similarly for the points Q_j.

(ii) Γ contains $\Gamma(N)$ for some composite odd square-free positive integer N such that $g_\Gamma \geq 2$.

(iii) The quotient moduli problem $[\Gamma(N)]^{\mathrm{can}}/H$ on $(\mathrm{Ell}/\mathbb{Z}[\zeta_N]^{\det(H)})$ is representable by a scheme $\mathcal{X}_\Gamma/\mathbb{Z}[\zeta_N]^{\det(H)}$.

We then have

$$\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathbb{Z}[\zeta_N]^{\det(H)}} = \frac{2g_\Gamma(V_0^\Gamma, V_\infty^\Gamma)_{\mathrm{fin}} - (V_0^\Gamma, V_0^\Gamma)_{\mathrm{fin}} - (V_\infty^\Gamma, V_\infty^\Gamma)_{\mathrm{fin}}}{2(g_\Gamma - 1)} - 2g_\Gamma(g_\Gamma - 1)\sum_{\sigma : K \hookrightarrow \mathbb{C}} g_{\mathrm{can}}^{\Gamma}(0^\sigma, \infty^\sigma),$$

where V_0^Γ and V_∞^Γ are distinguished vertical divisors of $\mathcal{X}_\Gamma$ associated to the cusps 0 and ∞ of $X(\Gamma)$ and $(\cdot,\cdot)_{\mathrm{fin}}$ stands for the usual intersection pairing on arithmetic surfaces. We call the first resp. second term on the right hand side the geometric and analytic contribution, respectively. In particular, the subgroup $\Gamma(N)$, under some restrictions on N, satisfies the conditions (i)–(iii) and thus we have the following theorem.

Theorem. *Let $\Gamma = \Gamma(N)$ with N a composite odd square-free positive integer and denote by $\mathcal{X}(N)/\mathbb{Z}[\zeta_N]$ the scheme representing the moduli problem of canonical $\Gamma(N)$-structures of elliptic curves. Then, as the level N tends to infinity, the following asymptotic expansion holds*

$$\frac{1}{\varphi(N)}\overline{\omega}^2_{\mathcal{X}(N)/\mathbb{Z}[\zeta_N]} = 4g_{\Gamma}\log(N) + o(g_{\Gamma}\log(N)).$$

In the general case, the approach for computing the analytic contribution of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathbb{Z}[\zeta_N]^{\det(H)}}$ is given by the spectral interpretation of the canonical Green's function $g^{\Gamma}_{\mathrm{can}}$ given by

$$g^{\Gamma}_{\mathrm{can}}(q_1,q_2) = 4\pi\mathscr{C}^{\Gamma}_{q_1q_2} + \frac{4\pi}{v_{\Gamma}} - 4\pi\left(\mathscr{R}^{\Gamma}_{q_1} + \mathscr{R}^{\Gamma}_{q_2}\right) + O\left(\frac{1}{g_{\Gamma}}\right),$$

where v_{Γ} denotes the hyperbolic volume of a fundamental domain of Γ, $\mathscr{R}^{\Gamma}_q$ is the constant term in the Laurent expansion at $s=1$ of the Ranking–Selberg transform at the cusp q of the Arakelov metric $F_{\Gamma}(z)$, and $\mathscr{C}^{\Gamma}_{q_1q_2}$ is the so-called scattering constant of Γ at the cusps q_1, q_2. In view of our guiding philosophy, we have the following proposition.

Theorem. *Let q_1, q_2 be two cusps of $X(\Gamma)$ having widths $w^{\Gamma}_{q_1}$ and $w^{\Gamma}_{q_2}$ respectively. Let $q_{1,1},\ldots,q_{1,r}$ denote all the cusps of $X(\Gamma(N))$ which are Γ-equivalent to q_1. Suppose that $q_{2,1}$ is a cusp of $X(\Gamma(N))$ that is Γ-equivalent to q_2. Then the following identity holds*

$$\mathscr{C}^{\Gamma}_{q_1q_2} = v_{\Gamma}^{-1}\log\left(\frac{N^2}{w^{\Gamma}_{q_1}w^{\Gamma}_{q_2}}\right) + \frac{N}{w^{\Gamma}_{q_1}}\sum_{j=1}^{r}\mathscr{C}^{\Gamma(N)}_{q_{1,j}q_{2,1}},$$

where r is the positive integer uniquely determined by the equality

$$r = (Nv_{\Gamma})^{-1}w^{\Gamma}_{q_1}v_{\Gamma(N)}.$$

Remark. It seems difficult to obtain results for $\mathscr{R}^{\Gamma}_q$ in the spirit of the previous proposition. However, if $\Gamma = \Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$, then computing the constants $\mathscr{R}^{\Gamma}_{0^\sigma}$ and $\mathscr{R}^{\Gamma}_{\infty^\sigma}$ amounts to computing only $\mathscr{R}^{\Gamma}_{\infty}$ and for this, we have

$$\mathscr{R}^{\Gamma}_{\infty} = -\frac{v_{\Gamma}^{-1}}{2g_{\Gamma}}\lim_{s\to1}\left(\frac{Z'_{\Gamma}}{Z_{\Gamma}}(s) - \frac{1}{s-1}\right) + \frac{1-\log(4\pi)}{4\pi g_{\Gamma}} + \frac{\mathscr{C}^{\Gamma}_{\infty\infty}}{g_{\Gamma}} + \mathscr{P}_{\Gamma} + \mathscr{E}_{\Gamma};$$

where Z_{Γ} is the Selberg zeta function and $\mathscr{P}_{\Gamma}$ and $\mathscr{E}_{\Gamma}$ denote the contributions coming from the parabolic and elliptic elements of Γ, respectively.

The approach for computing the geometric contribution of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathbb{Z}[\zeta_N]^{\det(H)}}$ is given by the morphism $\pi : \mathcal{X}_{\Gamma(N)} \longrightarrow \mathcal{X}_{\Gamma}$ coming from the moduli description of the scheme $\mathcal{X}_{\Gamma}/\mathbb{Z}[\zeta_N]^{\det(H)}$. In view of our guiding philosophy, the natural step to be taken is to consider the pullback of the distinguished divisors V_0^Γ and V_∞^Γ via the morphism π. In this manner, we reduce our original problem to intersections in the scheme $\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N]$. For the success of this approach, we need to parametrize the set of irreducible components of the fibers $(\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N])_{\mathfrak{p}}$ with $\mathfrak{p}|N$.

Theorem. *Let N be a composite odd square-free positive integer and suppose that $\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z}[\zeta_N])$ such that $\mathfrak{p}|p$ with $p|N$. Then the $p+1$ irreducible components of the fiber $(\mathcal{X}_{\Gamma(N)})_{\mathfrak{p}}$ are parametrized by $\mathbb{P}^1(\mathbb{F}_p)$. Furthermore, each of these components has multiplicity one.*

Outline of the thesis

Throughout the thesis we will state the main results for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$ although the first two cases can be found in the literature. For the proofs we adopt the following policy: For the key results we provide either full proofs or a sketch, for other results we refer to the literature.

In Chapter 1, we review the spectral theory of automorphic forms and emphasize the spectral expansion of automorphic kernels of weights 0 and 2. We also introduce the canonical Green's function and provide the spectral and automorphic interpretations of it.

In Chapter 2, we compute the scattering constants $\mathscr{C}^\Gamma_{q_1 q_2}$ for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$ at certain pairs of different cusps. In view of our guiding philosophy, we first determine the scattering constants for the subgroup $\Gamma(N)$ and then we derive a formula relating the scattering constants of $\Gamma_0(N)$ and $\Gamma_1(N)$ with those of $\Gamma(N)$.

In Chapter 3, we introduce the notion of an arithmetic surface and review the construction of the relative dualizing sheaf. Then we define certain moduli problems of elliptic curves associated to the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$. Under some conditions on the level N, each of these moduli problems is representable by an arithmetic surface $\mathcal{X}_\Gamma/\mathcal{O}_K$ whose generic fiber is isomorphic to the modular curves $X_0(N)/\mathbb{Q}$, $X_1(N)/\mathbb{Q}(\zeta_N)$, and $X(N)/\mathbb{Q}(\zeta_N)$ respectively. By means of Arakelov theory, we define the self-intersection of the relative dualizing sheaf $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ and use the theorems of Manin–Drinfeld, Falting–Hriljac, and Néron–Tate to obtain explicit expressions for the analytic and geometric contributions.

In Chapter 4, we proceed to the calculus of the analytic contribution of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$. By the computations of Chapter 2 and the spectral interpretation of the canonical Green's function, the calculus of the analytic contribution is reduced to determining the constant $\mathscr{R}^\Gamma_\infty$. To compute this constant, we use the spectral expansion of the automorphic kernels of weights 0 and 2 introduced in Chapter 1 and in analogy to the Selberg trace formula, we determine the contributions coming from the elliptic, hyperbolic, and parabolic matrices of Γ separately.

Finally, in Chapter 5, we describe a class of congruence subgroups Γ for which a formula for the self-intersection of the relative dualizing sheaf can be obtained by the same means as those in Chapter 3. Then we go back to the model $\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N]$ and parametrize the irreducible components of the fibers $(\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N])_{\mathfrak{p}}$ with $\mathfrak{p}|N$. As a result, we indicate how the geometric contribution of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ can be determined using a natural morphism $\pi : \mathcal{X}_{\Gamma(N)} \longrightarrow \mathcal{X}_\Gamma$ and test this method for $\Gamma = \Gamma_1(N)$. In the second half of this chapter, we analyze the asymptotic behaviour of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}/[K : \mathbb{Q}]$ as the level N tends to infinity, for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$.

Terminology and notation

Let $\mathbb{N} \coloneqq \{1, 2, \ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural, integer, rational, real, and complex numbers, respectively. For a given complex number $z = x+iy$ where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$, we refer to x and y as the x, y-coordinates of z. The real number x resp. y is called the real and the imaginary part of z, respectively; denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

Given a 2×2-matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{R}$, the determinant resp. trace of γ is defined by $\det(\gamma) \coloneqq ad - bc$ and $\mathrm{tr}(\gamma) \coloneqq a + d$, respectively. For all what follows, we set the 2×2 matrix

$$I \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Riemann zeta, Euler totient, and Möbius functions will be denoted by $\zeta(s)$, $\varphi(n)$, and $\mu(n)$, respectively. The function $\sigma_s(n)$ denotes the sum of the s-power of positive divisors of n, i.e., we have

$$\sigma_s(n) \coloneqq \sum_{d|N} d^s,$$

where $s \in \mathbb{C}$. For simplicity, we denote $\sigma_0(n)$ by $d(n)$ and $\sigma_1(n)$ by just $\sigma(n)$. The arithmetic function $\omega(n)$ counts the number of distinct prime factors of n.

Chapter 1

Analytic theory of automorphic forms

In this chapter we review the spectral theory of automorphic forms and introduce one of the central objects of this thesis: the canonical Green's function. As well, we fix terminology and notation that will be used all along.

The main references for this part are [AU97], [Hej83], [Iwa02], [Kat92], [Miy06], [Roe66], and [Roe67].

1.1 The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$

For the following considerations, let $\mathbb{P}^1(\mathbb{C}) \coloneqq \mathbb{C} \cup \{\infty\}$ be the Riemann sphere.

Definition 1.1.1. The *Poincaré half-plane or upper half-plane* is the subset $\mathbb{H}$ of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ given by

$$\mathbb{H} \coloneqq \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

equipped with the hyperbolic line element $\mathrm{d}s^2_{\mathrm{hyp}}(z)$, which in the x, y-coordinates is given by $\mathrm{d}s^2_{\mathrm{hyp}}(z) = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$.

The hyperbolic line element $\mathrm{d}s^2_{\mathrm{hyp}}(z)$ induces in a natural way the *hyperbolic distance* $d_{\mathrm{hyp}}(z, w)$ on $\mathbb{H}$, namely, for $z, w \in \mathbb{H}$ we set

$$d_{\mathrm{hyp}}(z, w) \coloneqq \inf_{\alpha} \left(\int_{\alpha} \mathrm{d}s^2_{\mathrm{hyp}}(z) \right)^{1/2};$$

here, the infimum runs over all continuous paths $\alpha : [0, 1] \longrightarrow \mathbb{H}$ with endpoints $\alpha(0) = z$ and $\alpha(1) = w$. Furthermore, an explicit formula for $d_{\mathrm{hyp}}(z, w)$ is

$$\cosh(d_{\mathrm{hyp}}(z, w)) = 1 + 2u(z, w),$$

where

$$u(z,w) := \frac{|z-w|^2}{4\,\mathrm{Im}(z)\mathrm{Im}(w)}. \tag{1.1}$$

The *hyperbolic volume form* $\mu_{\mathrm{hyp}}(z)$ on $\mathbb{H}$, in the x,y-coordinates, is given by

$$\mu_{\mathrm{hyp}}(z) := \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2},$$

and with this, the *hyperbolic volume* of a measurable subset $\mathscr{F} \subset \mathbb{H}$ is defined as the integral

$$\mathrm{vol}_{\mathrm{hyp}}(\mathscr{F}) := \int_{\mathscr{F}} \mu_{\mathrm{hyp}}(z).$$

In the sequel, we will write $\mathrm{d}x\,\mathrm{d}y/y^2$ instead of $\mathrm{d}x \wedge \mathrm{d}y/y^2$.

Definition 1.1.2. The *real projective line* is the subset $\mathbb{P}^1(\mathbb{R})$ of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ given by

$$\mathbb{P}^1(\mathbb{R}) := \mathbb{R} \cup \{\infty\}.$$

Note that the topological boundary of $\mathbb{H}$ is precisely given by the real projective line $\mathbb{P}^1(\mathbb{R})$.

Remark 1.1.3. Let $[x:y]$ denote the equivalence class of pairs $(x,y) \in \mathbb{R}^2$ with $(x,y) \neq (0,0)$ modulo the equivalence relation: $(x_1,y_1) \sim (x_2,y_2)$ if and only if there is a real number $\lambda \neq 0$ such that $x_1 = \lambda x_2$ and $y_1 = \lambda y_2$. Then using the homogeneous coordinates on $\mathbb{P}^1(\mathbb{R})$, we identify a point $x \in \mathbb{R}$ with the class $[x:1]$ and ∞ with the class $[1:0]$.

Consider the group of matrices

$$\mathrm{SL}_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}.$$

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ as follows: for a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and a point $z \in \mathbb{P}^1(\mathbb{C})$, set

$$\gamma z := \begin{cases} \dfrac{az+b}{cz+d}, & z \in \mathbb{C} \text{ and } z \neq -d/c; \\[2ex] \infty, & c \neq 0 \text{ and } z = -d/c; \text{ or } c = 0 \text{ and } z = \infty; \\[2ex] \dfrac{a}{c}, & c \neq 0 \text{ and } z = \infty. \end{cases} \tag{1.2}$$

The subsets $\mathbb{H}$ and $\mathbb{P}^1(\mathbb{R})$ of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ remain invariant under the action of $\mathrm{SL}_2(\mathbb{R})$; hence, (1.2) induces an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ and on $\mathbb{P}^1(\mathbb{R})$ which, in addition, turns out to be transitive.

Remark 1.1.4. For a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and a point $[x : y] \in \mathbb{P}^1(\mathbb{R})$ the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R})$, in the homogeneous coordinates, is given by

$$\gamma\,[x : y] \coloneqq [ax + by : cx + dy]\,. \tag{1.3}$$

Remark 1.1.5. Consider the transitive action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$. Then for any matrix $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and any point $z \in \mathbb{H}$, we have

$$\mathrm{d}s^2_{\mathrm{hyp}}(\gamma z) = \mathrm{d}s^2_{\mathrm{hyp}}(z), \qquad \mu_{\mathrm{hyp}}(\gamma z) = \mu_{\mathrm{hyp}}(z).$$

In other words, the hyperbolic line element $\mathrm{d}s^2_{\mathrm{hyp}}(z)$ and the hyperbolic volume form $\mu_{\mathrm{hyp}}(z)$ are invariant under the action of $\mathrm{SL}_2(\mathbb{R})$.

Let $\gamma \in \mathrm{SL}_2(\mathbb{R})$. The map from $\mathbb{H}$ onto itself given by $z \longmapsto \gamma z$ is called a *linear fractional transformation.* The set of all linear fractional transformations has a group structure under composition of maps, and it coincides with the group $\mathrm{Aut}(\mathbb{H})$ of all complex analytic automorphisms of $\mathbb{H}$ that preserve the orientation.

The correspondence that assigns to each matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ the linear fractional transformation $z \longmapsto (az + b)/(cz + d)$ is a surjective group homomorphism between $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{Aut}(\mathbb{H})$, whose kernel equals $\{\pm I\}$. Therefore, we have the group isomorphism

$$\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm I\}.$$

In the sequel, we will denote the quotient group $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ by $\mathrm{PSL}_2(\mathbb{R})$.

Notation 1.1.6. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be any subset. The image in $\mathrm{PSL}_2(\mathbb{R})$ of Γ will be denoted by $\overline{\Gamma}$.

Remark 1.1.7. Note that if $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a subgroup, then we have

$$\overline{\Gamma} \simeq \Gamma/(\Gamma \cap \{\pm I\}). \tag{1.4}$$

From now on, let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ be an arbitrary subgroup acting on the space $\mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$ via (1.2).

Definition 1.1.8. Let $z \in \mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$. The set $\Gamma z \coloneqq \{\gamma z \mid \gamma \in \Gamma\}$ is called the Γ-*orbit* of the point z. Two points $z, w \in \mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$ are Γ-*equivalent* if they belong to the same Γ-orbit; if this is not the case, we say that z and w are

Γ-*inequivalent*. The *space of orbits* $\Gamma\backslash\mathbb{H}$ is defined by $\Gamma\backslash\mathbb{H} := \{\Gamma z \mid z \in \mathbb{H}\}$.

Definition 1.1.9. A point $z \in \mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$ is a *fixed point* of a matrix $\gamma \in \Gamma$, if the equality $\gamma z = z$ holds. The *stabilizer of z in* Γ, denoted by Γ_z, consists of all matrices of Γ fixing z, i.e., we have $\Gamma_z := \{\gamma \in \Gamma \mid \gamma z = z\}$.

A simple verification of the quadratic equation in the variable z coming from the fixed point equation $\gamma z = z$, where $\gamma \in \Gamma$ and $\gamma \neq \pm I$, provides the following equivalences

$$\begin{aligned}
&\gamma \text{ has exactly one fixed point on } \mathbb{P}^1(\mathbb{R}) &&\Leftrightarrow \quad |\mathrm{tr}(\gamma)| = 2;\\
&\gamma \text{ has exactly two fixed points on } \mathbb{P}^1(\mathbb{R}) &&\Leftrightarrow \quad |\mathrm{tr}(\gamma)| > 2;\\
&\gamma \text{ has exactly one fixed point on } \mathbb{H} &&\Leftrightarrow \quad |\mathrm{tr}(\gamma)| < 2.
\end{aligned}$$

This motivates the following classification of matrices of $\mathrm{SL}_2(\mathbb{R})$ different from $\pm I$.

Definition 1.1.10. A matrix $\gamma \neq \pm I$ of $\mathrm{SL}_2(\mathbb{R})$ is called

(i) *parabolic*, if γ has exactly one fixed point on $\mathbb{P}^1(\mathbb{R})$;

(ii) *hyperbolic*, if γ has exactly two fixed points on $\mathbb{P}^1(\mathbb{R})$;

(iii) *elliptic*, if γ has exactly one fixed point on $\mathbb{H}$.

Definition 1.1.11. A point $z \in \mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$ is called *parabolic*, *hyperbolic*, or *elliptic fixed point with respect to* Γ, or simply *parabolic*, *hyperbolic*, or *elliptic point of* Γ, if it is a fixed point of a parabolic, hyperbolic, or elliptic matrix in Γ, respectively.

Notation 1.1.12. In the sequel, E_Γ resp. C_Γ will denote a complete set of representatives of Γ-inequivalent elliptic and parabolic points of Γ, respectively. The elements of E_Γ resp. C_Γ will be called *elliptic points of* $\Gamma\backslash\mathbb{H}$ and *cusps of* $\Gamma\backslash\mathbb{H}$, respectively. In addition, we set $e_{\Gamma} := \#E_\Gamma$ and $c_{\Gamma} := \#C_\Gamma$.

1.2 Fuchsian subgroups of the first kind

For the following considerations, we endow $\mathrm{SL}_2(\mathbb{R})$ with the topology inherited from $\mathbb{R}^4$ when $\mathrm{SL}_2(\mathbb{R})$ is identified with the set $\left\{(a,b,c,d) \in \mathbb{R}^4 \mid ad - bc = 1\right\}$.

Suppose that $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete topological subgroup. The discreteness of Γ can be formulated in terms of its action on $\mathbb{H}$. Indeed, recall that $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ acts *properly discontinuously on* $\mathbb{H}$ if for any two compact subsets $K_1, K_2 \subset \mathbb{H}$, the set $\{\gamma \in \Gamma \mid \gamma K_1 \cap K_2 \neq \emptyset\}$ is finite. Then we have that Γ is discrete if and only if Γ acts properly discontinuously on $\mathbb{H}$ (see [Kat92, Theorem 2.2.6, p. 32]).

Definition 1.2.1. A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is called *Fuchsian subgroup* if it is discrete.

Proposition 1.2.2. *Let $z \in \mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$ and $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a Fuchsian subgroup. Then the following assertions hold:*

(a) The stabilizer Γ_z of an elliptic point z of Γ satisfies $\Gamma_z \simeq \mathbb{Z}/n\mathbb{Z}$ for some integer $n > 1$, i.e., Γ_z is a finite cyclic group.

(b) The stabilizer Γ_z of a parabolic point z of Γ satisfies $\Gamma_z/(\Gamma \cap \{\pm I\}) \simeq \mathbb{Z}$. Furthermore, any element of Γ_z is either parabolic or equal to $\pm I$.

Proof. For the proof we refer the reader to [Miy06, Theorem 1.5.4, p. 18]. □

Definition 1.2.3. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup. The *order of $z \in \mathbb{H}$ with respect to* Γ is defined by $\mathrm{ord}_\Gamma(z) := \#\Gamma_z/\Gamma \cap \{\pm I\}$.

Definition 1.2.4. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup. A connected subset $\mathcal{F}_\Gamma \subset \mathbb{H}$ is called a *fundamental domain of* Γ if it satisfies the following conditions:

(i) $\mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma \mathcal{F}_\Gamma$;

(ii) $\mathcal{F}_\Gamma = \mathrm{cl}(U)$, where $\mathrm{cl}(\cdot)$ denotes the topological closure and U is an open set containing all the interior points of $\mathcal{F}_\Gamma$;

(iii) $\gamma U \cap U = \emptyset$, for all $\gamma \in \Gamma$ with $\gamma \neq \pm I$, where U is the open set given in (ii).

Definition 1.2.5. A Fuchsian subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a *Fuchsian subgroup of the first kind* if for some fundamental domain $\mathcal{F}_\Gamma$ of Γ, we have $\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_\Gamma) < \infty$.

Note that by [Kat92, Theorem 3.1.1, p. 50], the previous definition is independent of the choice of the fundamental domain $\mathcal{F}_\Gamma$.

Notation 1.2.6. In the sequel, if $\mathcal{F}_\Gamma$ is a fundamental domain of a Fuchsian subgroup of the first kind Γ, then we will write

$$v_\Gamma := \mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_\Gamma).$$

Proposition 1.2.7. *Let* $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ *be a Fuchsian subgroup of the first kind. Then we have* $e_\Gamma, c_\Gamma < \infty$.

Proof. For the proof we refer the reader to [Miy06, Theorem 1.7.8, p. 27]. □

Proposition 1.2.8. *Let* $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ *be a Fuchsian subgroup of the first kind. Suppose that* $\Gamma' \subset \Gamma$ *is a subgroup of finite index in* Γ *with coset decomposition* $\Gamma = \bigcup_{j=1}^{n} \Gamma' \gamma_j$. *Then the subset* $\mathcal{F}_{\Gamma'}$ *of* $\mathbb{H}$ *given by*

$$\mathcal{F}_{\Gamma'} := \bigcup_{j=1}^{n} \gamma_j \mathcal{F}_\Gamma$$

is a fundamental domain of Γ'. *Furthermore, we have*

$$v_{\Gamma'} = \begin{cases} (n/2) \cdot v_\Gamma, & \text{if } -I \in \Gamma \setminus \Gamma'; \\ n \cdot v_\Gamma, & \text{otherwise.} \end{cases}$$

Consequently, any subgroup of finite index of a Fuchsian subgroup of the first kind is also Fuchsian subgroup of the first kind.

Proof. For the proof we want to apply [Kat92, Theorem 3.1.2, p. 51], but note that this theorem is valid for subgroups of $\mathrm{PSL}_2(\mathbb{R})$. To remedy this situation, observe that the index $[\overline{\Gamma} : \overline{\Gamma'}]$ is equal to $[\Gamma : \Gamma']$ in either of the following cases: $-I \notin \Gamma$ or $-I \in \Gamma'$. In addition, $[\overline{\Gamma} : \overline{\Gamma'}]$ is equal to $(1/2)[\Gamma : \Gamma']$, if $-I \in \Gamma \setminus \Gamma'$. Then the result follows. □

We now consider the set

$$\mathbb{H}^* := \mathbb{H} \sqcup P_\Gamma,$$

where P_Γ denotes the set of parabolic points of Γ. Then $\mathbb{H}^*$ can be endowed with a topology such that $\mathbb{H}^*$ becomes a Hausdorff space (see [Miy06, p. 25]). Moreover, if we let Γ act on $\mathbb{H}^*$, then it can be proved that the quotient

$$\Gamma \backslash \mathbb{H}^* = (\Gamma \backslash \mathbb{H}) \sqcup C_\Gamma$$

is a Hausdorff space containing $\Gamma \backslash \mathbb{H}$ as an open subspace. In fact, $\Gamma \backslash \mathbb{H}^*$ is a Riemann surface. The interested reader is referred to [Miy06, §1.7 and §1.8] for further details.

Proposition 1.2.9. *Let* $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ *be a Fuchsian subgroup of the first kind. Then* $\Gamma \backslash \mathbb{H}^*$ *is a compact connected Riemann surface.*

Proof. For the proof we refer the reader to the proof of [Miy06, Theorem 1.9.1, pp. 32–35]. □

Notation 1.2.10. In the sequel, $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ will denote a Fuchsian subgroup of the first kind unless otherwise stated. The compact Riemann surface $\Gamma\backslash\mathbb{H}^*$ will be denoted by $X(\Gamma)$ and is called the *complex analytic curve of* Γ.

Finally, we will introduce the notion of a scaling matrix of a cusp.

Notation 1.2.11. For the sequel, we set the matrix

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

with $x \in \mathbb{R}$. Given an integer $M \geq 1$, define the set $B(M) := \{n(bM) \mid b \in \mathbb{Z}\}$. In case $M = 1$, we will simply write $B := B(1)$.

Definition 1.2.12. A *scaling matrix* of a cusp q of $\Gamma\backslash\mathbb{H}$ is a matrix $\sigma_q \in \mathrm{SL}_2(\mathbb{R})$ satisfying $\sigma_q\infty = q$ and $\{\pm I\} \cdot \sigma_q^{-1}\Gamma_q\sigma_q = \{\pm\, n(b) \mid b \in \mathbb{Z}\}$.

Remark 1.2.13. A scaling matrix σ_q of the cusp q is uniquely determined up to right multiplication by a matrix $\pm n(x)$ with $x \in \mathbb{R}$.

1.3 Automorphic functions and forms

In this section, we let k be a fixed non-negative even integer, $z = x + iy$ a point of $\mathbb{H}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a matrix of $\mathrm{SL}_2(\mathbb{R})$, and $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a Fuchsian subgroup of the first kind.

The *automorphy factor of weight* k is given by

$$j_\gamma(z; k) := \left(\frac{cz + d}{|cz + d|}\right)^k.$$

It can be verified that for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})$ the following identity holds

$$j_{\gamma_1\gamma_2}(z; k) = j_{\gamma_1}(\gamma_2 z; k) j_{\gamma_2}(z; k).$$

On the $\mathbb{C}$-vector space of functions $f : \mathbb{H} \longrightarrow \mathbb{C}$, we define the so-called *slash operator* $\cdot|[\gamma; k]$ as follows

$$(f|[\gamma; k])(z) := j_\gamma(z; k)^{-1} f(\gamma z).$$

A direct calculation shows that for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})$ the following identity holds

$$f|[\gamma_1\gamma_2; k] = (f|[\gamma_1; k])|[\gamma_2; k].$$

Definition 1.3.1. A map $f : \mathbb{H} \longrightarrow \mathbb{C}$ is an *automorphic function of weight k with respect to* Γ, or simply an *automorphic function of weight k*, if for all $\gamma \in \Gamma$, we have

$$(f|[\gamma; k])(z) = f(z).$$

Notation 1.3.2. The set of automorphic functions of weight k with respect to Γ will be denoted by $\mathscr{F}_k(\Gamma)$.

Let $f \in \mathscr{F}_k(\Gamma)$ and suppose that $q \in C_\Gamma$ is a cusp of $\Gamma\backslash\mathbb{H}$ with scaling matrix σ_q. Since either $n(1)$ or $-n(1)$ belongs to $\sigma_q^{-1}\Gamma\sigma_q$ and $\mathscr{F}_k(\Gamma)|[\delta; k] = \mathscr{F}_k(\delta^{-1}\Gamma\delta)$ holds for $\delta \in \mathrm{SL}_2(\mathbb{R})$, the function $(f|[\sigma_q; k])(z)$ is 1-periodic. Consequently, the function $f \in \mathscr{F}_k(\Gamma)$ possesses a Fourier series expansion for each cusp q of $\Gamma\backslash\mathbb{H}$.

Now, we proceed to describe a class of automorphic functions of weight k whose Fourier coefficients can be written in terms of the so-called *Whittaker's (second) function* $W_{\kappa;\mu}(t)$ (see Appendix A.4). First, let us introduce the following differential operators.

In the x, y-coordinates, the *hyperbolic Laplacian of weight k* is given by

$$\Delta_{\mathrm{hyp},k} := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$$

and the *Maass operators of weight k* are given by

$$\begin{aligned} \mathrm{K}_k &:= \frac{k}{2} + iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ \Lambda_k &:= \frac{k}{2} + iy\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\,. \end{aligned} \tag{1.5}$$

These three operators are related via the following identities

$$\Delta_{\mathrm{hyp},k} = \Lambda_{k+2}\mathrm{K}_k - \frac{k}{2}\left(1 + \frac{k}{2}\right) = \mathrm{K}_{k-2}\Lambda_k + \frac{k}{2}\left(1 - \frac{k}{2}\right). \tag{1.6}$$

Furthermore, the behaviour of the Maass operators K_k and Λ_k under the slash operator $\cdot|[\gamma; k]$ is described by

$$\begin{aligned} \mathrm{K}_k(f|[\gamma; k]) &= (\mathrm{K}_k f)|[\gamma; k+2], \\ \Lambda_k(f|[\gamma; k]) &= (\Lambda_k f)|[\gamma; k-2]. \end{aligned} \tag{1.7}$$

Using (1.6) and (1.7), we can conclude that the hyperbolic Laplacian $\Delta_{\mathrm{hyp},k}$ of weight k commutes with the slash operator $\cdot|[\gamma; k]$, i.e., we have

$$\Delta_{\mathrm{hyp},k}(f|[\gamma; k]) = (\Delta_{\mathrm{hyp},k} f)|[\gamma; k].$$

For what follows, let λ be a complex number and consider the differential equation

$$\Delta_{\mathrm{hyp},k} f - \lambda f = 0. \tag{1.8}$$

Proposition 1.3.3. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind and $q \in C_\Gamma$ a cusp of $\Gamma \backslash \mathbb{H}$ with scaling matrix σ_q. Suppose that $f \in \mathscr{F}_k(\Gamma)$ satisfies (1.8) for some $\lambda = 1/4 + r^2$ with $r \in \mathbb{C}$. Then we have*

$$(f|[\sigma_q; k])(z) = a_0(y, r, q; k) + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} a_n(z, r, q; k),$$

where

$$a_0(y, r, q; k) = \begin{cases} b_0(q; k) y^{\frac{1}{2} - ir} + c_0(q; k) y^{\frac{1}{2} + ir}, & r \neq 0; \\ b_0(q; k) y^{\frac{1}{2}} + c_0(q; k) y^{\frac{1}{2}} \log(y), & r = 0; \end{cases}$$

and

$$\begin{aligned} a_n(z, r, q; k) = \Big(& b_n(q; k) W_{-\frac{k}{2}\mathrm{sgn}(n); ir}(-4\pi |n| y) \\ & + c_n(q; k) W_{\frac{k}{2}\mathrm{sgn}(n); ir}(4\pi |n| y) \Big) e^{2\pi i n x}, \end{aligned}$$

for $n \neq 0$. Here, $b_n(q; k), c_n(q; k) \in \mathbb{C}$ and $W_{\kappa;\mu}(\cdot)$ denotes the Whittaker's (second) function (see Appendix A.4).

Proof. For the proof we refer the reader to [Roe66, pp. 300–301]. Note that, in Roelcke's notation, we have $m = 1$ and $v = \mathrm{id}$; hence, $\mu = 1$ and $\tau_\mu = 0$. □

To end this section, we will see how the (holomorphic) cusp forms of weight 2 from the classical theory of modular forms arise naturally in this context. First, let us recall the notion of a (holomorphic) cusp form of weight k.

Definition 1.3.4. A *holomorphic cusp form of weight k with respect to* Γ, or simply a *cusp form of weight k*, is a holomorphic function $g : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying the following properties:

(i) $g(\gamma z) = (cz + d)^k g(z)$, for $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma$;

(ii) $g(z) \mathrm{Im}(z)^{k/2}$ is bounded on $\mathbb{H}$.

Notation 1.3.5. The $\mathbb{C}$-vector space of (holomorphic) cusp forms of weight k with respect to Γ will be denoted by $\mathcal{S}_k(\Gamma)$.

Definition 1.3.6. An *automorphic form of weight k with respect to Γ and eigenvalue λ*, or simply an *automorphic form of weight k*, is a complex-valued function f defined on $\mathbb{H}$ satisfying the following properties:

(i) $f \in \mathscr{F}_k(\Gamma)$;

(ii) f is a real-analytic solution of (1.8);

(iii) for each $q \in C_\Gamma$ with scaling matrix σ_q, the asymptotic behaviour

$$(f|[\sigma_q;k])(z) = O(y^{\theta_q})$$

holds as $y \to \infty$, uniformly in x, for a suitable constant $\theta_q \in \mathbb{R}$ with $\theta_q \geq 0$.

Notation 1.3.7. The $\mathbb{C}$-vector space of all automorphic forms of weight k with respect to Γ and eigenvalue λ will be denoted by $\mathscr{A}_{k,\lambda}(\Gamma)$.

Remark 1.3.8. Note that if $f \in \mathscr{A}_{k,\lambda}(\Gamma)$, then for $n \neq 0$, the n-th Fourier coefficient $a_n(z,r,q;k)$ in Proposition 1.3.3 simplifies to

$$a_n(z,r,q;k) = c_n(q;k) W_{\frac{k}{2}\mathrm{sgn}(n);ir}(4\pi|n|y)e^{2\pi inx}.$$

This is a consequence of (iii) in Definition 1.3.6 and the asymptotic behaviour of Whittaker's function $W_{\kappa;\mu}(t) \sim t^{\kappa}e^{-t/2}$, as $|t| \to \infty$.

Definition 1.3.9. An *automorphic cusp form of weight k with respect to* Γ, or simply an *automorphic cusp form of weight k*, is an automorphic form $f \in \mathscr{A}_{k,\lambda}(\Gamma)$ satisfying $a_0(y,r,q;k) = 0$ for all $q \in C_\Gamma$.

Notation 1.3.10. The subspace of $\mathscr{A}_{k,\lambda}(\Gamma)$ consisting of all automorphic cusp forms of weight k will be denoted by $\mathscr{S}_{k,\lambda}(\Gamma)$.

With the previous definition, we can now state the following correspondence.

Proposition 1.3.11. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind. Then the map $f \longmapsto y^{-1}f$ defines a one to one correspondence between $\mathscr{S}_{2,0}(\Gamma)$ and $\mathcal{S}_2(\Gamma)$.*

Proof. For the proof we refer the reader to [Roe67, Satz 13.8 c), p. 323]. □

1.4 Eisenstein series of weight k

In this section, we let k be a fixed non-negative even integer, $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a Fuchsian subgroup of the first kind, and $q \in C_\Gamma$ a cusp of $\Gamma\backslash\mathbb{H}$ with scaling matrix σ_q. Set $\gamma_q \coloneqq \sigma_q n(1)\sigma_q^{-1} \in \mathrm{SL}_2(\mathbb{R})$ and $G_q \coloneqq \langle\gamma_q\rangle \subset \mathrm{SL}_2(\mathbb{R})$.

Definition 1.4.1. The *Eisenstein series of weight k with respect to Γ at the cusp q* of $\Gamma\backslash\mathbb{H}$, or simply *Eisenstein series of weight k at q*, is defined by

$$E^{\Gamma}_{q,k}(z,s) \coloneqq \frac{1}{2} \sum_{\gamma \in G_q\backslash\{\pm I\}\Gamma} j_{\sigma_q^{-1}\gamma}(z;k)^{-1} \operatorname{Im}(\sigma_q^{-1}\gamma z)^s, \tag{1.9}$$

where $z \in \mathbb{H}$, $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, and $j_{\sigma_q^{-1}\gamma}(z;k)$ is the automorphy factor of weight k given in Section 1.3.

Remark 1.4.2. In the literature, the Eisenstein series are usually defined for subgroups of $\mathrm{PSL}_2(\mathbb{R})$ (see, e.g., [Iwa02]). As it can be easily verified, the previous definition coincides with the literature's definition. For example, if we consider $k = 0$, then we have that (1.9) is equal to the Eisenstein series given by [Iwa02, (3.11), p. 57].

Remark 1.4.3. The Eisenstein series (1.9) is considered, e.g., in [Hej83, (5.19), p. 368] and [Roe67, (10.1), p. 291]. In the first reference, one has to take $r = 1$, $m = k$, $v = \mathrm{id}$, $\chi = \mathrm{id}$, $\alpha_{hj} = 0$, and $\vec{f}_{hj} = 1$ in [Hej83, Definition 5.3, p. 355], whereas in the second reference the assumption $-I \in \Gamma$ is implicit (see [Roe66, 6., p. 296]).

Lemma 1.4.4. *Let $s \in \mathbb{C}$, $\operatorname{Re}(s) > 1$. Then we have $E^{\Gamma}_{q,k}(z,s) \in \mathscr{A}_{k,s(1-s)}(\Gamma)$.*

Proof. For the proof we refer the reader to [Roe67, (10.10), p. 292]. □

Let $q_1, q_2 \in C_\Gamma$ be two cusps of $\Gamma\backslash\mathbb{H}$ with scaling matrices σ_{q_1} and σ_{q_2}, respectively. By the previous lemma and Remark 1.3.8, the Fourier series expansion of the Eisenstein series $E^{\Gamma}_{q_1,k}(z,s)$ of weight k at q_2 can be written as

$$(E^{\Gamma}_{q_1,k}(\cdot,s)|[\sigma_{q_2};k])(z) = a_0(y,s,q_2;k) + \sum_{n\neq 0} c_n(q_2;k) W_{\frac{k}{2}\operatorname{sgn}(n);ir}(4\pi|n|y)e^{2\pi i n x}.$$

Moreover, it can be proved that

$$a_0(y,s,q_2;k) = \delta_{q_1q_2} y^s + \varphi^{\Gamma}_{q_1q_2,k}(s) y^{1-s}, \tag{1.10}$$

where $\delta_{q_1q_2}$ denotes the Kronecker delta and $\varphi^{\Gamma}_{q_1q_2,k}(s)$ is a holomorphic function on the half-plane $\operatorname{Re}(s) > 1$ (see [Roe67, Lemma 10.2, p. 294]).

Definition 1.4.5. The *scattering function of weight k with respect to Γ at the cusps q_1, q_2*, or simply the *scattering function of weight k at q_1, q_2*, is the function $\varphi^{\Gamma}_{q_1q_2,k}(s)$ given by (1.10) defined on the half-plane $\operatorname{Re}(s) > 1$.

Notation 1.4.6. In the sequel, if the weight $k = 0$, then $E^{\Gamma}_{q,0}(z,s)$ resp. $\varphi^{\Gamma}_{q_1q_2,0}(s)$ will be denoted by $E^{\Gamma}_{q}(z,s)$ and $\varphi^{\Gamma}_{q_1q_2}(s)$, respectively.

Remark 1.4.7. Let $s \in \mathbb{C}$ of the form $s = 1/2 + ir$ with $r \in \mathbb{C}$. In the Fourier expansion

$$E^{\Gamma}_{q_1}(\sigma_{q_2}z, s) = \delta_{q_1q_2}y^s + \varphi^{\Gamma}_{q_1q_2}(s)y^{1-s} + \sum_{n\neq 0} c_n(q_2;0)W_{0;ir}(4\pi|n|y)e^{2\pi i n x}$$

of the Eisenstein series $E^{\Gamma}_{q_1}(z,s)$, the coefficients $c_n(q_2;0)$ are bounded by $|c_n(q_2;0)| \leq C(\varepsilon)e^{\varepsilon|n|}$ for any $\varepsilon > 0$, where $C(\varepsilon)$ is a constant depending only on ε. Hence, as $y \to \infty$, we have (see [Iwa02, Theorem 3.1, p. 54])

$$E^{\Gamma}_{q_1}(\sigma_{q_2}z, s) = \delta_{q_1q_2}y^s + \varphi^{\Gamma}_{q_1q_2}(s)y^{1-s} + O(e^{-2\pi y}).$$

Lemma 1.4.8. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a given Fuchsian subgroup of the first kind and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. For $r \in \mathbb{R}$, we set*

$$S^{\Gamma}_{q_1q_2}(r) = \#\left\{ d \bmod |r| \;\middle|\; \begin{pmatrix} * & * \\ r & d \end{pmatrix} \in \sigma^{-1}_{q_1}\Gamma\sigma_{q_2} \right\}. \tag{1.11}$$

Then the following assertions hold:

(a) If $-I \in \Gamma$, then the scattering function $\varphi^{\Gamma}_{q_1q_2}(s)$ is given by

$$\varphi^{\Gamma}_{q_1q_2}(s) = \sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \sum_{r>0} \frac{1}{r^{2s}}\, S^{\Gamma}_{q_1q_2}(r). \tag{1.12}$$

(b) If $-I \notin \Gamma$, then the scattering function $\varphi^{\Gamma}_{q_1q_2}(s)$ is given by

$$\varphi^{\Gamma}_{q_1q_2}(s) = \sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \sum_{r\neq 0} \frac{1}{r^{2s}}\, S^{\Gamma}_{q_1q_2}(r). \tag{1.13}$$

Proof. For the proof, the aim is to look for a double coset decomposition of $\sigma^{-1}_{q_1}\overline{\Gamma}\sigma_{q_2}$ with the condition that all the matrix representatives belong to $\sigma^{-1}_{q_1}\Gamma\sigma_{q_2}$.

First of all, note that for a given $L \in \sigma^{-1}_{q_1}\overline{\Gamma}\sigma_{q_2}$, which is not a translation, and any matrix $\gamma_L \in \mathrm{SL}_2(\mathbb{R})$ representing L, we have that either γ_L or $-\gamma_L$ belongs to $\sigma^{-1}_{q_1}\Gamma\sigma_{q_2}$. Then we have two cases:

(i) If γ_L and $-\gamma_L$ both belong to $\sigma^{-1}_{q_1}\Gamma\sigma_{q_2}$, then we choose the one having positive lower-left entry, say $\left(\begin{smallmatrix} a & b \\ \mu & d \end{smallmatrix}\right)$ with $\mu > 0$; therefore, we have that L belongs to the double class $B\left(\begin{smallmatrix} * & * \\ \mu & * \end{smallmatrix}\right)B$.

(ii) If say, $\gamma_L = \left(\begin{smallmatrix} a & b \\ \mu' & d \end{smallmatrix}\right)$ belongs to $\sigma^{-1}_{q_1}\Gamma\sigma_{q_2}$, then $L \in B\left(\begin{smallmatrix} * & * \\ \mu' & * \end{smallmatrix}\right)B$, where μ' could be negative.

It can be verified that case (i) leads to the following decomposition

$$\sigma^{-1}_{q_1}\overline{\Gamma}\sigma_{q_2} = \delta_{q_1q_2}B\omega B \cup \bigcup_{r>0}\bigcup_{d \bmod r} B\omega_{rd}B,$$

where ω and ω_{rd} are matrices in $\sigma_{q_1}^{-1}\Gamma\sigma_{q_2}$ given by $\omega = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $\omega_{rd} = \left(\begin{smallmatrix} a & * \\ r & d \end{smallmatrix}\right)$. Similarly, the case (ii) gives the decomposition

$$\sigma_{q_1}^{-1}\overline{\Gamma}\sigma_{q_2} = \delta_{q_1q_2}B\omega B \cup \bigcup_{r\neq 0}\ \bigcup_{d \bmod |r|} B\omega_{rd}B.$$

The lemma follows by using these decompositions in the computation of the Fourier coefficients of $E_{q_1}^{\Gamma}(\sigma_{q_2}z, s)$. This concludes the proof. □

Remark 1.4.9. If $-I \in \Gamma$ and $r > 0$, then $S_{q_1q_2}^{\Gamma}(r)$ is equal to the Kloosterman sum $\mathcal{S}_{q_1q_2}(0, 0; r)$ (see [Iwa02, (2.24), p. 48]). Note that loc. cit. $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$.

In the next proposition we state some of the most important properties for Eisenstein series of weight k.

Proposition 1.4.10. *Let $q \in C_\Gamma$ be a cusp of $\Gamma\backslash\mathbb{H}$. Then the following assertions hold:*

(a) The Eisenstein series $E_{q,k}^{\Gamma}(z, s)$ of weight k at q has a meromorphic continuation to the whole s-plane.

(b) The meromorphically continued Eisenstein series of part (a) has finitely many poles in the half-plane $\mathrm{Re}(s) > 1/2$. *These poles are all simple and belong to the interval $(\frac{1}{2}, 1]$. In particular, if $k = 0$, then the point $s = 1$ is always a pole whose residue is equal to v_Γ^{-1}.*

(c) The meromorphically continued Eisenstein series of part (a) is holomorphic on the line $\mathrm{Re}(s) = 1/2$.

(d) The functional equation

$$E_{q,k}^{\Gamma}(z, s) = \sum_{\tilde{q}\in C_\Gamma} \varphi_{q\tilde{q},k}^{\Gamma}(s)E_{\tilde{q},k}^{\Gamma}(z, 1 - s)$$

holds for the meromorphically continued Eisenstein series of part (a) provided that s and $1 - s$ are not poles.

Proof. For the proof of part (a) we refer the reader to [Hej83, §11, p. 108], for parts (b) and (c) we refer to [Roe67, Satz 10.3, p. 297], [Roe67, Satz 10.4, p. 299], and [Iwa02, Proposition 6.13, p. 91]; and for part (d) see [Roe67, (10.18), p. 296]. □

Proposition 1.4.10 implies, in particular, that the scattering function $\varphi_{q_1q_2}^{\Gamma}(s)$ extends to a meromorphic function on $\mathrm{Re}(s) \geq 1/2$ such that the point $s = 1$ is always a simple pole with residue equal to v_Γ^{-1}.

Definition 1.4.11. The *scattering constant* $\mathscr{C}^{\Gamma}_{q_1q_2}$ *of* Γ *at* $q_1, q_2 \in C_\Gamma$ is the constant term in the Laurent expansion of $\varphi^{\Gamma}_{q_1q_2}(s)$ at $s = 1$, i.e., we have

$$\mathscr{C}^{\Gamma}_{q_1q_2} := \lim_{s\to 1}\left(\varphi^{\Gamma}_{q_1q_2}(s) - \frac{v_\Gamma^{-1}}{s-1}\right). \tag{1.14}$$

Finally, we gather some useful identities for Eisenstein series that will be needed in later chapters.

Lemma 1.4.12. *Let* $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ *be a Fuchsian subgroup of the first kind and* k *a non-negative even integer. Then the following assertions hold:*

(a) The Eisenstein series $E^{\Gamma}_q(z,s)$ *at a cusp* q *of* $\Gamma\backslash\mathbb{H}$ *satisfies*

$$E^{\Gamma}_q(\gamma z, s) = E^{\gamma^{-1}\Gamma\gamma}_{\gamma^{-1}q}(z,s),$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$*.*

(b) The image of the meromorphically continued Eisenstein series $E^{\Gamma}_{q,k}(z,s)$ *of weight* k *at* q *under the operator* K_k *is given by*

$$\mathrm{K}_k\Big(E^{\Gamma}_{q,k}(z,s)\Big) = \left(\frac{k}{2}+s\right)E^{\Gamma}_{q,k+2}(z,s),$$

where s *is not a pole of* $E^{\Gamma}_{q,k}(z,s)$*.*

Proof. The proof of part (a) follows from a direct calculation. For part (b) we refer the reader to [Roe67, (10.8), p. 292] and the first paragraph of [Roe67, p. 294]. □

Lemma 1.4.13. *Let* $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ *be a Fuchsian subgroup of the first kind and* k *a non-negative even integer. Suppose that* $\infty \in C_\Gamma$ *is a cusp of* $\Gamma\backslash\mathbb{H}$*. Then we have*

$$\sum_{q\in C_\Gamma}\left|a_0\Big(y, \frac{1}{2}+ir, q; k\Big)\right|^2 = 2y + y^{1+2ir}\varphi^{\Gamma}_{\infty\infty}\Big(\frac{1}{2}-ir\Big)\left(\frac{\frac{1}{2}+ir}{\frac{1}{2}-ir}\right)^{k/2} + y^{1-2ir}\varphi^{\Gamma}_{\infty\infty}\Big(\frac{1}{2}+ir\Big)\left(\frac{\frac{1}{2}-ir}{\frac{1}{2}+ir}\right)^{k/2},$$

for $k = 0, 2$*, where* $a_0(y,s,q;k)$ *denotes the 0-th Fourier coefficient of* $E^{\Gamma}_{\infty,k}(z,s)$ *at* q*.*

Proof. For the proof we refer the reader to [May14, (5.1),(5.2), pp. 135–136]. □

1.5 Spectral expansions

In this section, we let k be a non-negative even integer, $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a Fuchsian subgroup of the first kind, and $\mathcal{F}_\Gamma$ a fixed fundamental domain. In addition, we set the integral

$$\langle f, g\rangle := \int\limits_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\mu_{\mathrm{hyp}}(z). \tag{1.15}$$

First of all, let us introduce the spectral expansion of an automorphic function of weight k. Let $\mathscr{H}_k(\Gamma)$ be the Hilbert space of measurable functions $f \in \mathscr{F}_k(\Gamma)$ satisfying $\langle f, f\rangle < \infty$, endowed with the inner product $\langle f, g\rangle_{k,\Gamma} := \langle f, g\rangle$. The hyperbolic Laplacian $\Delta_{\mathrm{hyp},k}$ of weight k defines an operator

$$\Delta_{\mathrm{hyp},k} : \mathscr{D}_k^2 \longrightarrow \mathscr{H}_k(\Gamma), \tag{1.16}$$

where $\mathscr{D}_k^2$ is the dense subset of $\mathscr{H}_k(\Gamma)$ given by

$$\mathscr{D}_k^2 := \{f \in \mathscr{H}_k(\Gamma) \cap \mathscr{C}_k^2(\Gamma) \mid \Delta_{\mathrm{hyp},k} f \in \mathscr{H}_k(\Gamma)\};$$

here, $\mathscr{C}_k^2(\Gamma)$ denotes the set of twice continuously differentiable functions $f \in \mathscr{F}_k(\Gamma)$ in the variables $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ with $z \in \mathbb{H}$.

Roelcke proved that (1.16) is in fact a symmetric operator (see [Roe66, p. 309]). Using a criterion for essential self-adjointness of symmetric operators from general spectral theory, we have the following result.

Lemma 1.5.1. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind and k a non-negative even integer. Then the operator $\Delta_{\mathrm{hyp},k}$ given by* (1.16) *is symmetric and essentially self-adjoint.*

Proof. For the proof we refer the reader to [Roe66, Satz 3.2, p. 309]. □

The importance of the previous lemma lies in the fact that (1.16) can be extended to a unique self-adjoint operator $\widetilde{\Delta}_k$ (defined on a suitable domain). Consequently, we have the following proposition.

Proposition 1.5.2. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind and k a non-negative even integer. Then there exists a countable orthonormal set $\{\psi_j\}_{j=0}^{\infty}$ of eigenfunctions of $\widetilde{\Delta}_k$ such that for all $f \in \mathscr{H}_k(\Gamma)$ we have*

$$f(z) = \sum_{j=0}^{\infty} \langle f, \psi_j\rangle_{k,\Gamma}\, \psi_j(z) + \sum_{q \in C_\Gamma} \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} \Big\langle f, E_{q,k}^{\Gamma}\Big(\cdot, \frac{1}{2} + ir\Big)\Big\rangle E_{q,k}^{\Gamma}\Big(z, \frac{1}{2} + ir\Big) \mathrm{d}r$$

which converges in the norm topology. Furthermore, if f is smooth and bounded, then the expansion converges uniformly on compacta of $\mathbb{H}$.

Proof. For the proof we refer the reader to [Roe66, Satz 5.7, p. 325], [Roe67, Satz 12.1, p. 309], [Roe67, Satz 12.2, p. 316], and [Roe67, Satz 12.3, p. 318]. $\square$

One of the implications of Proposition 1.5.2 which will be important for our purposes is the spectral decomposition of the so-called automorphic kernels of weight k, which we now proceed to describe.

Notation 1.5.3. For the sequel, let $h : \mathbb{R} \longrightarrow \mathbb{C}$ denote the function given by

$$h(r) := e^{-t\left(\frac{1}{4}+r^2\right)}, \tag{1.17}$$

where $t > 0$ is a fixed real number. Furthermore, let ϕ_k denotes the *inverse Selberg/Harish–Chandra transform of weight k* of $h(r)$ (see Appendix B.1).

Definition 1.5.4. The *point-pair invariant of weight k* associated to h is the function $\pi_k : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{C}$ given by

$$\pi_k(z, w) := \left(\frac{w - \bar{z}}{z - \bar{w}}\right)^{k/2} \phi_k(u(z, w)), \tag{1.18}$$

where $u(z, w)$ is the map given by (1.1).

Definition 1.5.5. The *automorphic kernel of weight k with respect to* Γ, or simply *automorphic kernel of weight k*, is given by

$$\mathcal{K}_k^\Gamma(z, w) := \frac{1}{2} \sum_{\gamma \in \{\pm I\}\Gamma} j_\gamma(w; k)\pi_k(z, \gamma w).$$

Remark 1.5.6. The automorphic kernel $\mathcal{K}_k^\Gamma(z, w)$ of weight k belongs to the Hilbert space $\mathscr{H}_k(\Gamma)$ (see [Hej83, (6.11), p. 387]).

Theorem 1.5.7. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind. Suppose that $\{\psi_j\}_{j=1}^\infty$ is a countable orthonormal set of eigenfunctions of $\widetilde{\Delta}_0$ with corresponding eigenvalues $\lambda_j = \frac{1}{4} + r_j^2$, for some $r_j \in \mathbb{C}$, and $\{f_j\}_{j=1}^{g_\Gamma}$ is an orthonormal basis of $\mathcal{S}_2(\Gamma)$. Then the following identities hold*

$$\mathcal{K}_0^\Gamma(z, w) = v_\Gamma^{-1} + \sum_{j=1}^\infty h(r_j)\psi_j(z)\overline{\psi_j}(w)$$
$$+ \frac{1}{4\pi} \sum_{q \in C_\Gamma} \int_{-\infty}^\infty h(r) E_q^\Gamma\left(z, \frac{1}{2} + ir\right) \overline{E_q^\Gamma\left(w, \frac{1}{2} + ir\right)} \mathrm{d}r,$$

$$\mathcal{K}_2^\Gamma(z,w) = \sum_{j=1}^{g_\Gamma} \mathrm{Im}(z)\mathrm{Im}(w) f_j(z)\overline{f_j}(w) + \sum_{j=1}^{\infty} \frac{h(r_j)}{\lambda_j} (\mathrm{K}_0\psi_j)(z)\overline{(\mathrm{K}_0\psi_j)}(w)$$
$$+ \frac{1}{4\pi} \sum_{q\in C_\Gamma} \int_{-\infty}^{\infty} h(r) E_{q,2}^\Gamma\Big(z, \frac{1}{2}+ir\Big) \overline{E_{q,2}^\Gamma\Big(w, \frac{1}{2}+ir\Big)} \mathrm{d}r.$$

Here, K_0 *is the Maass operator given by* (1.5)*.*

Proof. (Sketch.) Fix $w \in \mathbb{H}$ and apply Proposition 1.5.2 to $\mathcal{K}_0^\Gamma(z,w)$ and $\mathcal{K}_2^\Gamma(z,w)$. Next, unfold the integrals $\langle \mathcal{K}_k^\Gamma(\cdot,w), E_q^\Gamma(\cdot, 1/2+ir)\rangle$ and apply [Hej76, Proposition 2.14, p. 364]; thus we obtain

$$\Big\langle \mathcal{K}_0^\Gamma(\cdot,w), E_q^\Gamma\Big(\cdot, \frac{1}{2}+ir\Big)\Big\rangle = h(r)\overline{E_q^\Gamma\Big(w, \frac{1}{2}+ir\Big)},$$
$$\Big\langle \mathcal{K}_2^\Gamma(\cdot,w), E_{q,2}^\Gamma\Big(\cdot, \frac{1}{2}+ir\Big)\Big\rangle = h(r)\overline{E_{q,2}^\Gamma\Big(w, \frac{1}{2}+ir\Big)}.$$

Then the result follows. □

1.6 Canonical Green's function

Let X be a compact connected Riemann surface of genus $g_X \geq 1$. On X we consider the operators d and d^c given by $\mathrm{d} := \partial + \overline{\partial}$ and $\mathrm{d}^c := (4\pi i)^{-1}(\partial - \overline{\partial})$ verifying the following identities

$$\mathrm{d}\mathrm{d}^c = -(2\pi i)^{-1}\partial\overline{\partial},$$
$$-4\pi \mathrm{d}\mathrm{d}^c f(z) = (\Delta_{\mathrm{hyp}} f(z))\mu_{\mathrm{hyp}}(z),$$

where Δ_{hyp} stands for the hyperbolic Laplacian $\Delta_{\mathrm{hyp},0}$ of weight 0.

For the following considerations, let Ω_X^1 be the $\mathbb{C}$-vector space of dimension g_X consisting of global holomorphic 1-forms on X endowed with the inner product

$$\langle \alpha, \beta\rangle_{\Omega_X^1} := \frac{i}{2}\int_X \alpha \wedge \overline{\beta},$$

and let us fix $\{\alpha_j\}_{j=1}^{g_X}$, an orthonormal basis of Ω_X^1.

Definition 1.6.1. The *canonical volume form on* X is the $(1,1)$-form given by

$$\mu_{\mathrm{can}} := \frac{i}{2g_X}\sum_{j=1}^{g_X} \alpha_j \wedge \overline{\alpha}_j.$$

Remark 1.6.2. This definition is independent of the choice of the orthonormal basis $\{\alpha_j\}_{j=1}^{g_X}$. Moreover, it can be verified that μ_{can} is normalized in the sense that $\int_X \mu_{\mathrm{can}} = 1$.

If now $X = X(\Gamma)$ (see Notation 1.2.10) for $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a Fuchsian subgroup of the first kind of genus $g_\Gamma \geq 1$, then by the Kodaira–Spencer isomorphism we have $\mathcal{S}_2(\Gamma) \simeq \Omega_X^1$ (see [Miy06, Theorem 2.3.2]); hence, there is an orthonormal basis $\{f_j\}_{j=1}^{g_\Gamma}$ of $\mathcal{S}_2(\Gamma)$ that corresponds to $\{\alpha_j\}_{j=1}^{g_X}$. Therefore, the $(1,1)$-form μ_{can}, in the local coordinate z, is given by

$$\mu_{\mathrm{can}}(z) = \left(\frac{\mathrm{Im}(z)^2}{g_\Gamma} \sum_{j=1}^{g_\Gamma} |f_j(z)|^2 \right) \mu_{\mathrm{hyp}}(z).$$

Definition 1.6.3. The *canonical Green's function* $g_{\mathrm{can}}(z,w)$ is the unique real-valued smooth function defined on $(X \times X) \setminus \Delta_X$, where Δ_X is the diagonal of X, such that for each $w \in X$, the following properties are satisfied:

(i) $\mathrm{dd}^c g_{\mathrm{can}}(z,w) + \delta_w(z) = \mu_{\mathrm{can}}(z)$, where $\delta_w(\cdot)$ denotes the Dirac delta distribution;

(ii) $\displaystyle\int_X g_{\mathrm{can}}(z,w)\mu_{\mathrm{can}}(z) = 0.$

Notation 1.6.4. In the sequel, we will write $g_{\mathrm{can}}^\Gamma(z,w)$ if we want to emphasize that the underlying compact connected Riemann surface is of the form $X(\Gamma)$, for a Fuchsian subgroup of the first kind $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$.

Definition 1.6.5. The *Arakelov metric on* $X(\Gamma)$ is the function given by

$$F_\Gamma(z) := \frac{\mathrm{Im}(z)^2}{g_\Gamma} \sum_{j=1}^{g_\Gamma} |f_j(z)|^2, \tag{1.19}$$

where $z \in X(\Gamma)$ and $\{f_j\}_{j=1}^{g_\Gamma}$ is an orthonormal basis of the space $\mathcal{S}_2(\Gamma)$.

Remark 1.6.6. The Arakelov metric $F_\Gamma(z)$ on $X(\Gamma)$ is independent of the choice of the orthonormal basis $\{f_j\}_{j=1}^{g_\Gamma}$.

Now we proceed to describe the numerical invariant $\mathscr{R}_q^\Gamma$ associated to $F_\Gamma(z)$.

Definition 1.6.7. A function $f \in \mathscr{F}_0(\Gamma)$ is of *rapid decay at a cusp* $q \in C_\Gamma$ if the 0-th Fourier coefficient $a_0(y,q)$ of f at the cusp q satisfies $a_0(y,q) = O(y^{-M})$ for all $M > 0$, as $y \to \infty$.

Lemma 1.6.8. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind. Then $F_\Gamma(z)$ is of rapid decay at every cusp $q \in C_\Gamma$.*

Proof. For the proof, let $g_j(z) := \mathrm{Im}(z) f_j(z)$; thus $F_\Gamma(z) = (1/g_\Gamma) \sum_{j=1}^{g_\Gamma} |g_j(z)|^2$. Note that $g_j \in \mathscr{S}_{2,0}(\Gamma)$ by Proposition 1.3.11; therefore, for any cusp $q \in C_\Gamma$, we have $g_j|[\sigma_q; 2](z) = O(e^{-2\pi \mathrm{Im}(z)})$, as $\mathrm{Im}(z) \to \infty$ (see [Roe66, Lemma 2.1, p. 302]). The result follows since $|(g_j|[\sigma_q; 2])(z)|^2 = |g_j(\sigma_q z)|^2$ holds. $\square$

Definition 1.6.9. The *Rankin–Selberg transform at a cusp $q \in C_\Gamma$* of a function $f \in \mathscr{F}_0(\Gamma)$ of rapid decay at q is the integral

$$\mathcal{R}_q^\Gamma[f](s) := \int\limits_{\mathcal{F}_\Gamma} f(z) E_q^\Gamma(z,s) \mu_{\mathrm{hyp}}(z),$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

The function $\mathcal{R}_q^\Gamma[f](s)$ is holomorphic on the half-plane $\mathrm{Re}(s) > 1$. Moreover, it has a meromorphic continuation to the complex plane, with a simple pole at the point $s = 1$ whose residue equals $v_\Gamma^{-1} \cdot \int_{\mathcal{F}_\Gamma} f(z)\mu_{\mathrm{hyp}}(z)$. In particular, for the function $F_\Gamma(z)$, we have the identities

$$\int\limits_{\mathcal{F}_\Gamma} F_\Gamma(z)\mu_{\mathrm{hyp}}(z) = \int\limits_{\mathcal{F}_\Gamma} \mu_{\mathrm{can}}(z) = 1.$$

Definition 1.6.10. The *Rankin–Selberg constant $\mathscr{R}_q^\Gamma$ of $F_\Gamma(z)$ at $q \in C_\Gamma$* is the constant term in the Laurent expansion of $\mathcal{R}_q^\Gamma[F_\Gamma](s)$ at $s = 1$, i.e., we have

$$\mathscr{R}_q^\Gamma := \lim_{s \to 1} \left(\mathcal{R}_q^\Gamma[F_\Gamma](s) - \frac{v_\Gamma^{-1}}{s-1} \right). \tag{1.20}$$

Remark 1.6.11. The numerical invariant $\mathscr{R}_q^\Gamma$ as well as the scattering constant $\mathscr{C}_{q_1 q_2}^\Gamma$ defined in (1.14) are key in determining the value of the canonical Green's function evaluated at two different cusps (see Theorem 1.6.13).

Next, we define another kind of Green's function which is related with g_{can}^Γ.

Definition 1.6.12. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$. The *automorphic Green's function $G_s^\Gamma(z,w)$* is given by

$$G_s^\Gamma(z,w) := \frac{1}{4\pi} \sum_{\gamma \in \{\pm I\}\Gamma} Q_{s-1}(2u(z,\gamma w) + 1), \tag{1.21}$$

where $z, w \in \mathbb{H}$, $u(\cdot,\cdot)$ is given by (1.1), and $Q_\nu(u)$ denotes the *Legendre function of the second kind* (see Appendix A.3).

Theorem 1.6.13. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind and $q_1, q_2 \in C_\Gamma$ two different cusps of $\Gamma\backslash\mathbb{H}$. Suppose that $X(\Gamma)$ has genus $g_{\Gamma} \geq 1$. Then we have*

$$g_{\mathrm{can}}^{\Gamma}(q_1, q_2) = 4\pi \mathscr{C}_{q_1 q_2}^{\Gamma} + \frac{4\pi}{v_{\Gamma}} - 4\pi\Big(\mathscr{R}_{q_1}^{\Gamma} + \mathscr{R}_{q_2}^{\Gamma}\Big) + 4\pi \lim_{s\to 1}\Bigg(\int\limits_{X(\Gamma)\times X(\Gamma)} G_s^{\Gamma}(z,w)\mu_{\mathrm{can}}(z)\mu_{\mathrm{can}}(w) - \frac{v_{\Gamma}^{-1}}{s(s-1)}\Bigg),$$

where $\mathscr{C}_{q_1 q_2}^{\Gamma}$ is given by Definition 1.4.11, $\mathscr{R}_{q_j}^{\Gamma}$ by (1.20) with $j = 1, 2$, and $G_s^{\Gamma}(z,w)$ denotes the automorphic Green's function given by (1.21).

Proof. (Sketch.) This result is given in [AU97, Proposition E, p. 5]. Note that $g_{\mathrm{can}}^{\Gamma}(z,w) = -2g_{\mathrm{Ar}}(z,w)$ and $G_s^{\Gamma}(z,w) = -G_s(z,w)$, where g_{Ar} resp. $G_s(z,w)$ is the canonical Green's function and automorphic Green's function, respectively, considered in [AU97].

For the convenience of the reader, we sketch the proof. For this we write $X = X(\Gamma)$ and fix a smooth and symmetric real valued function g defined on $(X \times X) \setminus \Delta_X$ satisfying (i) of Definition 1.6.3 and the following normalization

$$\int\limits_{X\times X} g(z,w)\mu_{\mathrm{hyp}}(z)\mu_{\mathrm{hyp}}(w) = 0.$$

Then $g_{\mathrm{can}}^{\Gamma}$ and g differ by a constant. Using the spectral expansion of automorphic functions and standard methods of the spectral theory of automorphic forms, one can determine this constant (for details, see [AU97, pp. 7–20]). More precisely, we have

$$g_{\mathrm{can}}^{\Gamma}(z,w) - g(z,w) = 4\pi \lim_{s\to 1}\Bigg(\int\limits_{X\times X} G_s^{\Gamma}(z,w)\mu_{\mathrm{can}}(z)\mu_{\mathrm{can}}(w) - \frac{v_{\Gamma}^{-1}}{s(s-1)}\Bigg),$$

where $z, w \in X$ (see [AU97, Lemme 2.2.6, p. 20]). In particular, this formula holds for $(z,w) = (q_1, q_2)$. After four pages of computations (see [AU97, pp. 16–20]), it can be proved that $g(q_1, q_2)$ can be written in terms of the Arakelov metric $F_\Gamma(z)$ on X. More precisely, we have

$$g(q_1, q_2) = 4\pi \lim_{Y\to\infty}\lim_{s\to 1}\Bigg[\frac{v_{\Gamma}^{-1}}{s(s-1)} - \frac{Y^{2-2s}}{1-2s}\varphi_{q_1 q_2}^{\Gamma}(s) - \frac{Y^{1-s}}{2s-1}\Big(\mathcal{R}_{q_1}^{\Gamma}[F_\Gamma](s) + \mathcal{R}_{q_2}^{\Gamma}[F_\Gamma](s)\Big)\Bigg].$$

The result follows by first finding the Laurent expansion at $s = 1$ of the term in brackets and then taking the limits. □

Chapter 2

Eisenstein series and scattering constants for congruence subgroups

In this chapter we determine the scattering constants of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$ at the pairs of cusps given by the analytic contribution of the self-intersection of the relative dualizing sheaf, see (4.2). These formulas are well-known by the experts and can also be found in the literature (see, e.g., [Hej83, Chapter 11, pp. 532–568]).

For the convenience of the reader, we provide a full presentation of the material. We have organized the content of this chapter in such a way that it corresponds to our guiding philosophy: Invariants of congruence subgroups can be completely determined using data coming from $\Gamma(N)$.

In Section 2.1, we recall some aspects of the modular group. In particular, we concentrate on the Eisenstein series and the scattering function. Next we introduce the notion of width of a cusp $q \in C_\Gamma$, where Γ is a subgroup of finite index of the modular group.

In Section 2.2, we study the Eisenstein series at ∞ of the principal congruence subgroup $\Gamma(N)$ and determine certain scattering functions. Using the Laurent expansions of Appendix A, we provide explicit expressions for the corresponding scattering constants in terms of the level N.

In Section 2.3, we derive a formula for the scattering constant of an arbitrary congruence subgroup Γ at a pair of cusps $q_1, q_2 \in C_\Gamma$ using almost exclusively data coming from $\Gamma(N)$. With this result, we finally compute the remaining scattering constants for the subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$.

The main references for this part are [DS05], [Hej83], [Kat92], and [Miy06]. We also use some ideas and results of [Pos10] and [Kei06].

2.1 The case $\mathrm{SL}_2(\mathbb{Z})$

Notation 2.1.1. For the sequel, we set $\varrho := e^{2\pi i/3} \in \mathbb{H}$ and we let $\mathscr{C}$ be the constant given by

$$\mathscr{C} := 1 - \log(4\pi) + \frac{\zeta'(-1)}{\zeta(-1)}. \tag{2.1}$$

Definition 2.1.2. The *modular group* $\mathrm{SL}_2(\mathbb{Z})$ consists of all 2×2 matrices of determinant equal to one with integer entries, i.e., we have

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z},\, ad - bc = 1 \right\}.$$

For all what follows, we denote $\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ by $\mathrm{PSL}_2(\mathbb{Z})$.

Lemma 2.1.3. *Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then Γ is a Fuchsian subgroup of the first kind. Furthermore, we have $v_{\Gamma} = \pi/3$.*

Proof. For the proof we refer the reader to [Miy06, Theorem 4.1.2 (2), p. 97]. □

Lemma 2.1.4. *Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then the following assertions hold:*

(a) There are exactly two elliptic points of $\Gamma\backslash\mathbb{H}$. Furthermore, $E_\Gamma = \{i, \varrho\}$ is a complete set of representatives of the elliptic points of $\Gamma\backslash\mathbb{H}$ with stabilizers given by

$$\Gamma_i = \left\{ \pm I, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \text{ and } \Gamma_\varrho = \left\{ \pm I, \pm \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\}.$$

(b) There is exactly one cusp of $\Gamma\backslash\mathbb{H}$. Furthermore, $C_\Gamma = \{\infty\}$ is a complete set of representatives of the cusp of $\Gamma\backslash\mathbb{H}$ with stabilizer

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{Z} \right\}.$$

Proof. For the proof we refer the reader to [Miy06, Theorem 4.1.3, p. 98] and [Miy06, Lemma 1.3.2, p. 8]. □

Proposition 2.1.5. *Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Suppose that $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 1$. *Then the following assertions hold:*

(a) The Eisenstein series $E^{\Gamma}_\infty(z, s)$ of the modular group at the cusp $\infty \in C_\Gamma$,

given by Definition 1.4.1, satisfies

$$E_\infty^\Gamma(z,s) = \frac{1}{2\zeta(2s)} \sideset{}{'}\sum_{(m,n)\in\mathbb{Z}^2} \frac{y^s}{|mz+n|^{2s}},$$

where $'$ indicates that the sum runs over all pairs of integers different from $(0,0)$.

(b) *The scattering function $\varphi_{\infty\infty}^\Gamma(s)$ of the modular group at ∞ is given by*

$$\varphi_{\infty\infty}^\Gamma(s) = \sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}.$$

Furthermore, at $s=1$, we have the Laurent expansion

$$\varphi_{\infty\infty}^\Gamma(s) = \frac{3/\pi}{s-1} + \frac{6}{\pi}\mathscr{C} + O(s-1), \tag{2.2}$$

where $\mathscr{C}$ is the constant given by (2.1).

Proof. For the proof of part (a), let us choose $\sigma_\infty = I$ for the scaling matrix of ∞. Then we have $\gamma_\infty = \sigma_\infty n(1)\sigma_\infty^{-1} = n(1)$ and $G_\infty = \langle n(1)\rangle = B$. Since we have the bijection

$$B\backslash\Gamma \simeq \{(m,n)\in\mathbb{Z}^2 \mid \gcd(m,n)=1\},$$

given by the assignment $\left(\begin{smallmatrix} * & * \\ m & n\end{smallmatrix}\right) \longmapsto (m,n)$, we obtain from Definition 1.4.1, observing that $\{\pm I\}\Gamma = \Gamma$, the following identity

$$E_\infty^\Gamma(z,s) = \frac{1}{2}\sum_{\gamma\in B\backslash\Gamma} \operatorname{Im}(\gamma z)^s = \frac{1}{2}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}}.$$

Now, we have

$$\sideset{}{'}\sum_{(m,n)\in\mathbb{Z}^2} \frac{y^s}{|mz+n|^{2s}} = \sum_{d\geq 1}\sideset{}{'}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \gcd(m,n)=d}} \frac{y^s}{|mz+n|^{2s}} = \sum_{d\geq 1}\sideset{}{'}\sum_{\substack{(m',n')\in\mathbb{Z}^2\\ \gcd(m',n')=1}} \frac{y^s}{|dm'z+dn'|^{2s}}$$

$$= \zeta(2s)\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}}.$$

This yields part (a).

For the proof of part (b), note that by (1.12) of Lemma 1.4.8, we have

$$\varphi_{\infty\infty}^\Gamma(s) = \sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\sum_{c=1}^\infty \frac{1}{c^{2s}} S_{\infty\infty}^\Gamma(c),$$

where $S^{\Gamma}_{\infty\infty}(c)$ is given by (1.11). In particular, in our case, we have

$$S^{\Gamma}_{\infty\infty}(c) = \#\left\{ d \bmod c \;\middle|\; \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \right\}.$$

The set on the right hand side is in bijection with the set

$$\{d \in \mathbb{Z} \mid 1 \leq d \leq c,\, \gcd(c,d) = 1\}.$$

Hence, we obtain

$$S^{\Gamma}_{\infty\infty}(c) = \varphi(c).$$

Using the identity

$$\sum_{c=1}^{\infty} \frac{\varphi(c)}{c^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)}$$

which holds for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ (see [Apo76, Example 4, p. 229]), we obtain

$$\varphi^{\Gamma}_{\infty\infty}(s) = \sqrt{\pi}\, \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}.$$

Finally, from the Laurent expansions (A.1) and (A.2) of Appendix A, one can easily deduce

$$\varphi^{\Gamma}_{\infty\infty}(s) = \frac{3/\pi}{s-1} + \frac{6}{\pi}\mathscr{C} + O(s-1).$$

This concludes the proof. □

In what follows, we let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and $q \in C_\Gamma$ a cusp.

Lemma 2.1.6. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and $z \in \mathbb{H}$ an elliptic point of Γ. Then the order $\mathrm{ord}_\Gamma(z)$ of z with respect to Γ is either equal to 2 or 3.*

Proof. For the proof we refer the reader to [Miy06, Lemma 4.2.6, p. 107]. □

In view of the previous lemma, let us write the number e_{Γ} of elliptic points of $\Gamma\backslash\mathbb{H}$ as $e_{\Gamma} = \nu_2(\Gamma) + \nu_3(\Gamma)$, where

$$\nu_j(\Gamma) \coloneqq \#\{z \in E_\Gamma \mid \mathrm{ord}_\Gamma(z) = j\}. \tag{2.3}$$

Lemma 2.1.7. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index, c_{Γ} the number of cusps of $\Gamma\backslash\mathbb{H}$, and $\nu_j(\Gamma)$ as in (2.3) with $j = 2, 3$. Then the genus g_{Γ} of the* *complex analytic curve $X(\Gamma)$ is given by*

$$g_\Gamma = 1 + \frac{v_\Gamma}{4\pi} - \frac{\nu_2(\Gamma)}{4} - \frac{\nu_3(\Gamma)}{3} - \frac{c_\Gamma}{2}.$$

In particular, we have $g_{\mathrm{SL}_2(\mathbb{Z})} = 0$.

Proof. For the genus formula, we refer the reader to [Miy06, Theorem 4.2.11, p. 113]. The identity $g_{\mathrm{SL}_2(\mathbb{Z})} = 0$ is a direct consequence of Lemma 2.1.3 and Lemma 2.1.4. □

Notation 2.1.8. For $y \in \mathbb{R}$ with $y \neq 0$, we define the matrix

$$a(y) \coloneqq \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}.$$

Definition 2.1.9. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and $q \in C_\Gamma$. The *width w_q^Γ of the cusp q* is defined by

$$w_q^\Gamma \coloneqq [\mathrm{PSL}_2(\mathbb{Z})_q : \overline{\Gamma}_q].$$

Remark 2.1.10. Given a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of finite index, there is a natural covering map $f : X(\Gamma) \longrightarrow \mathbb{P}^1(\mathbb{C})$. In this case, the width w_q^Γ of the cusp q coincides with the ramification index of the covering map f at q (see [Shi94, Proposition 1.37, p. 20]).

Remark 2.1.11. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and $q \in C_\Gamma$. Choose $g_q \in \mathrm{SL}_2(\mathbb{Z})$ such that $g_q\infty = q$ and define the matrix

$$\sigma_q \coloneqq g_q \cdot a\Big((w_q^\Gamma)^{1/2}\Big). \tag{2.4}$$

Then, we have $\sigma_q\infty = g_q\infty = q$ and $\{\pm I\}\sigma_q^{-1}\Gamma_q\sigma_q = \{\pm n(b) \mid b \in \mathbb{Z}\}$, so (2.4) provides a scaling matrix of q.

Notation 2.1.12. In the sequel, given any subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of finite index, we choose for each cusp $q \in C_\Gamma$ a matrix $g_q \in \mathrm{SL}_2(\mathbb{Z})$ satisfying $g_q\infty = q$; therefore, in what follows scaling matrices will be of the form (2.4).

We end this section with the following lemma which will be useful for determining the scattering functions of the next section. For the sake of clarity, we write B_q to denote the set $B(w_q^\Gamma) = \{n(bw_q^\Gamma) \mid b \in \mathbb{Z}\}$.

Lemma 2.1.13. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and $q_1, q_2 \in C_\Gamma$ two cusps, not necessarily distinct, with corresponding widths $w_{q_1}^\Gamma$ and $w_{q_2}^\Gamma$,* *respectively. Suppose that $r \in \mathbb{R}$ is given. Then the following identity holds*

$$S^{\Gamma}_{q_1 q_2}(r) = \#\left\{ d \bmod |cw^{\Gamma}_2| \;\middle|\; \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in g_{q_1}^{-1}\Gamma g_{q_2} \right\},$$

where $c = r/(w^{\Gamma}_1 w^{\Gamma}_2)^{1/2}$. Here, $S^{\Gamma}_{q_1 q_2}(r)$ is given by (1.11) and g_{q_1}, g_{q_2} are given by Notation 2.1.12. In particular, if $r/(w_1 w_2)^{1/2}$ is an integer, then $S^{\Gamma}_{q_1 q_2}(r) > 0$.

Proof. For the proof, let us write $g_i := g_{q_i}$, $w_i := w^{\Gamma}_{q_i}$, $a_i := a(w_i)$, $\sigma_i := \sigma_{q_i}$, and $B_i := B_{q_i}$ $(i = 1, 2)$ for the sake of clarity.

Note that by the identity (1.11), the number $S^{\Gamma}_{q_1 q_2}(r)$ counts the number of double cosets of $B\backslash\sigma_1^{-1}\Gamma\sigma_2/B$ with lower-left entry equal to r. Then to prove the lemma it suffices to show that there exists a bijection between $B\backslash\sigma_1^{-1}\Gamma\sigma_2/B$ and $B_1\backslash g_1^{-1}\Gamma g_2/B_2$.

Consider the map

$$\varphi : B\backslash\sigma_1^{-1}\Gamma\sigma_2/B \longrightarrow B_1\backslash g_1^{-1}\Gamma g_2/B_2,$$

given by the assignment

$$B\nu B \longmapsto B_1\left(a_1 \nu a_2^{-1}\right)B_2,$$

where $\nu \in \sigma_1^{-1}\Gamma\sigma_2$. We claim that φ is a bijection. Indeed, to prove the surjectivity of φ, let $B_1\alpha B_2 \in B_1\backslash g_1^{-1}\Gamma g_2/B_2$ with $\alpha = g_1^{-1}\gamma g_2$, for some $\gamma \in \Gamma$. Then, we have

$$\begin{aligned} \varphi(B(\sigma_1^{-1}\gamma\sigma_2)B) &= B_1(a_1\sigma_1^{-1}\gamma\sigma_2 a_2^{-1})B_2 = B_1 g_1^{-1}\gamma g_2 B_2 \\ &= B_1\alpha B_2, \end{aligned}$$

where for the second equality we used the identities $\sigma_1^{-1} = a_1^{-1}g_1^{-1}$ and $\sigma_2 = g_2 a_2$. This proves the surjectivity of φ.

To prove the injectivity of φ, suppose that

$$B_1\left(a_1\alpha a_2^{-1}\right)B_2 = B_1\left(a_1\beta a_2^{-1}\right)B_2$$

with $\alpha, \beta \in \sigma_1^{-1}\Gamma\sigma_2$. Then we have

$$\begin{aligned} B\alpha B &= a_1^{-1}(B_1 a_1\alpha a_2^{-1}B_2)a_2 = a_1^{-1}(B_1 a_1\beta a_2^{-1}B_2)a_2 \\ &= B\beta B, \end{aligned}$$

where for the first and third equality we used the identity $B = a_i^{-1}B_i a_i$ $(i = 1, 2)$. This proves the injectivity of φ.

Note that if $\nu = \left(\begin{smallmatrix} * & * \\ r & d \end{smallmatrix}\right) \in \sigma_1^{-1}\Gamma\sigma_2$, then the matrix

$$a_1 \nu a_2^{-1} = \begin{pmatrix} * & * \\ r/(w_1w_2)^{1/2} & d/(w_1w_2)^{1/2} \end{pmatrix} \in g_1^{-1}\Gamma g_2.$$

Since $g_1^{-1}\Gamma g_2 \subset \mathrm{SL}_2(\mathbb{Z})$, we have that $r/(w_1w_2)^{1/2} \in \mathbb{Z}$. Furthermore, the double coset $\varphi(\nu) = \left(\begin{smallmatrix} * & * \\ c & \delta \end{smallmatrix}\right)$ of $B_1 \backslash g_1^{-1}\Gamma g_2^{\Gamma}/B_2$ has c entry fixed and equal to $r/(w_1w_2)^{1/2}$ and δ is a positive integer uniquely determined modulo $|cw_2|$.

Consequently, we have

$$S^{\Gamma}_{q_1q_2}(r) = \#\left\{ \delta \bmod |cw_2| \;\middle|\; \begin{pmatrix} * & * \\ c & \delta \end{pmatrix} \in g_1^{-1}\Gamma g_2 \right\}.$$

This concludes the proof. □

2.2 The case $\Gamma(N)$

For the sequel, let N be a positive integer.

Definition 2.2.1. The *principal congruence subgroup* $\Gamma(N)$ is defined by

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1 \bmod N,\ b \equiv c \equiv 0 \bmod N \right\}.$$

Lemma 2.2.2. *Let $N \geq 1$ be an integer. Then the principal congruence subgroup $\Gamma(N)$ is a normal subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. Furthermore, if $N \geq 3$, then we have*

$$v_{\Gamma(N)} = \frac{\pi}{6} N^2 \varphi(N) \prod_{p|N} \left(1 + \frac{1}{p}\right),$$

where the product runs over all prime numbers p dividing N. In particular, $\Gamma(N)$ is a Fuchsian subgroup of the first kind.

Proof. For the proof we refer the reader to [Miy06, Theorem 4.2.5, p. 106] and Proposition 1.2.8. □

Lemma 2.2.3. *Let $N \geq 3$ be an integer. Then the following assertions hold:*

(a) The principal congruence subgroup $\Gamma(N)$ has no elliptic points, that is, we have $e_{\Gamma(N)} = 0$.

(b) The number of cusps of $\Gamma(N)\backslash\mathbb{H}$ is given by

$$c_{\Gamma(N)} = \frac{1}{2} N \varphi(N) \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

(c) The genus $g_{\Gamma(N)}$ of $X(\Gamma(N))$ is given by

$$g_{\Gamma(N)} = 1 + \frac{v_{\Gamma(N)}}{4\pi}\left(1 - \frac{6}{N}\right).$$

Proof. For the proof of parts (a) and (b), we refer the reader to [Miy06, Theorem 4.2.10, p. 112]. Part (c) is an immediate consequence of (a) and (b) together with Lemma 2.1.7 and Lemma 2.2.2. □

Lemma 2.2.4. *Let $N \geq 3$ be an integer and $q \in C_{\Gamma(N)}$. Then we have*

$$w_q^{\Gamma(N)} = N. \tag{2.5}$$

Proof. For the proof, we claim that $w_\infty^{\Gamma(N)} = N$. Indeed, this follows from the identities $\mathrm{PSL}_2(\mathbb{Z})_\infty \simeq B$ and $\overline{\Gamma(N)}_\infty \simeq B(N)$. Now, since $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$, by Remark 2.1.10 and [Shi94, Proposition 1.37, p. 20], all the cusps q of $\Gamma(N)\backslash\mathbb{H}$ have the same width. This concludes the proof. □

Proposition 2.2.5. *Let $N \geq 3$ be an integer. Suppose that $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Then the following identity holds*

$$E_\infty^{\Gamma(N)}(z,s) = \frac{1}{N^s} \sum_{\substack{1 \leq u \leq N \\ \gcd(u,N)=1}} D_u(s) \cdot \left(\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\equiv(0,u) \bmod N}} \frac{y^s}{|mz+n|^{2s}} \right),$$

where

$$D_u(s) = \sum_{\substack{d=1 \\ du\equiv 1 \bmod N}}^{\infty} \frac{\mu(d)}{d^{2s}}.$$

Proof. For the proof we choose $g_\infty = I$; therefore, by (2.4) and (2.5), we obtain $\sigma_\infty = a(N^{1/2})$. This implies that $\gamma_\infty = \sigma_\infty n(1) \sigma_\infty^{-1} = n(N)$ and therefore, we have $G_\infty = \langle n(N) \rangle = B(N) = \Gamma(N)_\infty$.
Now, using the bijection (see [Miy06, Lemma 7.1.6, p. 273])

$$\Gamma(N)_\infty\backslash\Gamma(N) \simeq \{(m,n) \in \mathbb{Z}^2 \mid \gcd(m,n) = 1,\ (m,n) \equiv (0,1) \bmod N\}$$

in Definition 1.4.1, we obtain

$$\begin{aligned} E_\infty^{\Gamma(N)}(z,s) &= \frac{1}{2N^s} \sum_{\gamma\in\Gamma(N)_\infty\backslash\{\pm I\}\Gamma(N)} \mathrm{Im}(\gamma z)^s = \frac{1}{N^s} \sum_{\gamma\in\Gamma(N)_\infty\backslash\Gamma(N)} \mathrm{Im}(\gamma z)^s \\ &= \frac{1}{N^s} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1 \\ (m,n)\equiv(0,1) \bmod N}} \frac{y^s}{|mz+n|^{2s}}. \end{aligned}$$

By [Gol73, (32), p. 38], we have

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \gcd(m,n)=1\\ (m,n)\equiv(g,h)\bmod N}} \frac{y^s}{|mz+n|^{2s}} = \sum_{\substack{1\leq u\leq N\\ \gcd(u,N)=1}} D_u(s)\cdot\left(\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (m,n)\equiv(gu,hu)\bmod N}} \frac{y^s}{|mz+n|^{2s}}\right),$$

where $g,h\in\mathbb{Z}$ such that $\gcd(g,h,N)=1$. Evaluating this identity at $g=0$ and $h=1$, the assertion follows. $\square$

Now we proceed to the calculation of the scattering functions of $\Gamma(N)$ at pairs of distinguished cusps. Let us first establish some notation.

Notation 2.2.6. Let $v\in(\mathbb{Z}/N\mathbb{Z})^\times$ and v' an integer satisfying the congruence condition $vv'\equiv 1\bmod N$. We set $0_v := [1:v']\in\mathbb{P}^1(\mathbb{R})$ and

$$g_{0_v} := \begin{pmatrix} 1 & -x\\ v' & -mN\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}),$$

where $x,m\in\mathbb{Z}$ such that $\det(g_{0_v})=1$.

Remark 2.2.7. Note that 0_v is a cusp of $\Gamma(N)\backslash\mathbb{H}$ (see [DS05, p. 100]).

Lemma 2.2.8. *Let $N\geq 3$ be an odd square-free positive integer and suppose that $c\in\mathbb{Z}$. Then the following identity holds*

$$S_{\infty\infty}^{\Gamma(N)}(cN) = \begin{cases}\varphi(|c|N)/\varphi(N), & \text{if } c\equiv 0\bmod N;\\ 0, & \text{otherwise.}\end{cases}$$

Furthermore, if $v\in(\mathbb{Z}/N\mathbb{Z})^\times$, then we have

$$S_{0_v\infty}^{\Gamma(N)}(cN) = \begin{cases}\varphi(|c|), & \text{if } c\equiv -v'\bmod N;\\ 0, & \text{otherwise;}\end{cases}$$

where $v'\in\mathbb{Z}$ such that $vv'\equiv 1\bmod N$.

Proof. For the proof of the first identity, we choose $g_\infty = I$. Taking $\Gamma=\Gamma(N)$ and $q_1=q_2=\infty$ in Lemma 2.1.13 with $B_{q_1}=B_{q_2}=B(N)$ and $w_{q_1}^\Gamma = w_{q_2}^\Gamma = N$, we have

$$S_{\infty\infty}^{\Gamma(N)}(cN) = \#\left\{ d \bmod |c|N \;\middle|\; \begin{pmatrix} * & *\\ c & d\end{pmatrix}\in\;\Gamma(N)\right\},$$

From this, we note that $S_{\infty\infty}^{\Gamma(N)}(cN)>0$ provided that $c\equiv 0\bmod N$. Moreover, the right hand side is equal to

$$\#\left\{1 \leq d \leq |c|N \mid \gcd(|c|, d) = 1,\, d \equiv 1 \bmod N\right\}.$$

This cardinality is computed, e.g., in [Hej83, Lemma 5.7, p. 545] and is equal to $\varphi(|c|N)/\varphi(N)$. This yields the first identity.

For the proof of the second identity, we choose $g_\infty = I$ and g_{0_v} given by Notation 2.2.6. Taking $\Gamma = \Gamma(N)$, $q_1 = 0_v$, and $q_2 = \infty$ in Lemma 2.1.13 with $B_{q_1} = B_{q_2} = B(N)$ and $w_{q_1}^\Gamma = w_{q_2}^\Gamma = N$, we have

$$S_{0_v\infty}^{\Gamma(N)}(cN) = \#\left\{d \bmod |c|N \;\middle|\; \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in g_{0_v}^{-1}\Gamma(N)g_\infty\right\}.$$

Note that $g_{0_v}^{-1}\Gamma(N)g_\infty = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \left(\begin{smallmatrix} 0 & v \\ -v' & 1 \end{smallmatrix}\right) \bmod N\right\}$. Thus, we have $S_{0_v\infty}^{\Gamma(N)}(cN) > 0$ provided that $c \equiv -v' \bmod N$. Moreover the previous cardinality is equal to

$$\#\{1 \leq d \leq |c|N \mid \gcd(|c|, d) = 1,\, d \equiv 1 \bmod N\},$$

and for computing this cardinality, we use again [Hej83, Lemma 5.7, p. 545] and obtain $\varphi(|c|)$. This concludes the proof. □

Proposition 2.2.9. *Let $N \geq 3$ be an odd square-free positive integer and suppose that $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Then the following identity holds*

$$\varphi_{\infty\infty}^{\Gamma(N)}(s) = 2\sqrt{\pi}\,\frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}N^{1-4s}\prod_{p \mid N}\frac{1}{1 - p^{-2s}}.$$

Furthermore, if $v \in (\mathbb{Z}/N\mathbb{Z})^\times$, then we have

$$\varphi_{0_v\infty}^{\Gamma(N)}(s) = 2\sqrt{\pi}\,\frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}\frac{N^{-2s}}{\varphi(N)}\sum_{\substack{\chi \bmod N \\ \text{even}}}\overline{\chi}(v')\frac{L(2s-1,\chi)}{L(2s,\chi)},$$

where $v' \in \mathbb{Z}$ is such that $vv' \equiv 1 \bmod N$ and the sum runs over all even Dirichlet characters χ modulo N (see Appendix D.1), i.e., χ satisfies $\chi(-1) = 1$.

Proof. For the proof of the first identity, we take $q_1 = q_2 = \infty$ and choose $g_\infty = I$; then we obtain $\sigma_\infty = a(N^{1/2})$. By (1.11), we have

$$S_{\infty\infty}^{\Gamma(N)}(r) = \#\left\{d \bmod |r| \;\middle|\; \begin{pmatrix} * & * \\ r & d \end{pmatrix} \in a(N^{-1/2})\Gamma(N)a(N^{1/2})\right\}.$$

The condition $\left(\begin{smallmatrix} * & * \\ r & d \end{smallmatrix}\right) \in a(N^{-1/2})\Gamma(N)a(N^{1/2})$ implies that r is an integer of the form $r = cN$, where $c \in \mathbb{Z}$ such that $c \equiv 0 \bmod N$. Consequently, by virtue of part (b) in the Lemma 1.4.8, we have

$$\varphi_{\infty\infty}^{\Gamma(N)}(s) = \sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \sum_{\substack{c\in\mathbb{Z}\\ c\neq 0\\ c\equiv 0 \bmod N}} \frac{1}{(cN)^{2s}}\, S_{\infty\infty}^{\Gamma(N)}(cN).$$

Now, by applying Lemma 2.2.8, we obtain

$$\begin{aligned}
\varphi_{\infty\infty}^{\Gamma(N)}(s) &= \sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{N^{-2s}}{\varphi(N)} \sum_{\substack{c\in\mathbb{Z}\\ c\neq 0\\ c\equiv 0 \bmod N}} \frac{\varphi(|c|N)}{c^{2s}}\\
&= 2\sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{N^{-2s}}{\varphi(N)} \sum_{\substack{n=1\\ n\equiv 0 \bmod N}} \frac{\varphi(nN)}{n^{2s}}.
\end{aligned}$$

Since $\varphi(ab) = \varphi(a)\varphi(b)\cdot(d/\varphi(d))$ with $d = \gcd(a,b)$, we have

$$\varphi_{\infty\infty}^{\Gamma(N)}(s) = 2\sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{N^{1-2s}}{\varphi(N)} \sum_{\substack{n=1\\ n\equiv 0 \bmod N}}^{\infty} \frac{\varphi(n)}{n^{2s}}.$$

For convenience, we write the previous identity as follows

$$\varphi_{\infty\infty}^{\Gamma(N)}(s) = 2\sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{N^{1-4s}}{\varphi(N)} \sum_{m=1}^{\infty} \frac{\varphi(mN)}{m^{2s}}.$$

Now, for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, we have the identity

$$\frac{1}{N^{2s}}\sum_{m=1}^{\infty}\frac{\varphi(mN)}{m^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)}\prod_{p|N}\frac{p-1}{p^{2s}-1}. \tag{2.6}$$

This can be deduced, e.g., from lemmas 4.5 and 4.6 in [Hej83, p. 535] by taking $A_1 = A_2 = 1$. Using (2.6), we obtain

$$\begin{aligned}
\varphi_{\infty\infty}^{\Gamma(N)}(s) &= 2\sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} N^{1-2s}\left(\frac{\zeta(2s-1)}{\zeta(2s)}\prod_{p|N}\frac{1}{p^{2s}-1}\right)\frac{1}{\varphi(N)}\prod_{p|N}(p-1)\\
&= 2\sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} N^{1-4s}\frac{\zeta(2s-1)}{\zeta(2s)}\prod_{p|N}\frac{1}{1-p^{-2s}}.
\end{aligned}$$

The last equality follows because N is square-free.

For the proof of the second identity, we proceed similarly. We take $q_1 = 0_v$, $q_2 = \infty$ and choose g_{0_v} as in Notation 2.2.6 and $g_\infty = I$; then we obtain

$\sigma_{0_v} = \left(\begin{smallmatrix} \sqrt{N} & -x/\sqrt{N} \\ v'\sqrt{N} & -m\sqrt{N} \end{smallmatrix}\right)$ and $\sigma_\infty = a(N^{1/2})$. By (1.11), we have

$$S_{0_v\infty}^{\Gamma(N)}(r) = \#\left\{ d \bmod |r| \;\middle|\; \begin{pmatrix} * & * \\ r & d \end{pmatrix} \in \sigma_{0_v}^{-1}\Gamma(N)a(N^{1/2}) \right\}.$$

The condition $\left(\begin{smallmatrix} * & * \\ r & d \end{smallmatrix}\right) \in \sigma_{0_v}^{-1}\Gamma(N)a(N^{1/2})$ implies that r is an integer of the form $r = cN$, where $c \in \mathbb{Z}$. Consequently, by virtue of part (b) in the Lemma 1.4.8, we have

$$\varphi_{0_v\infty}^{\Gamma(N)}(s) = \sqrt{\pi}\, \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \sum_{\substack{c\in\mathbb{Z} \\ c\neq 0}} \frac{1}{(cN)^{2s}}\, S_{0_v\infty}^{\Gamma(N)}(cN).$$

By Lemma 2.2.8 we obtain

$$\begin{aligned} \varphi_{0_v\infty}^{\Gamma(N)} &= \sqrt{\pi}\, \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{1}{N^{2s}} \sum_{\substack{c\in\mathbb{Z} \\ c\neq 0 \\ c\equiv -v' \bmod N}} \frac{\varphi(|c|)}{c^{2s}} \\ &= \sqrt{\pi}\, \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{1}{N^{2s}} \Bigg(\sum_{\substack{n=1 \\ n\equiv v' \bmod N}}^{\infty} \frac{\varphi(n)}{n^{2s}} + \sum_{\substack{n=1 \\ n\equiv -v' \bmod N}}^{\infty} \frac{\varphi(n)}{n^{2s}} \Bigg). \end{aligned}$$

By the orthogonality relations of Dirichlet characters (see Appendix D.1), we have

$$\begin{aligned} \sum_{\substack{n=1 \\ n\equiv -v' \bmod N}}^{\infty} \frac{\varphi(n)}{n^{2s}} &= \sum_{n=1}^{\infty} \Bigg(\frac{1}{\varphi(N)} \sum_{\chi \bmod N} \chi(n)\overline{\chi}(-v) \Bigg) \frac{\varphi(n)}{n^{2s}}, \\ \sum_{\substack{n=1 \\ n\equiv v' \bmod N}}^{\infty} \frac{\varphi(n)}{n^{2s}} &= \sum_{n=1}^{\infty} \Bigg(\frac{1}{\varphi(N)} \sum_{\chi \bmod N} \chi(n)\overline{\chi}(v) \Bigg) \frac{\varphi(n)}{n^{2s}}, \end{aligned}$$

where the sum runs over all Dirichlet characters modulo N. Therefore, we obtain

$$\begin{aligned} \varphi_{0_v\infty}^{\Gamma(N)}(s) &= \sqrt{\pi}\, \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{N^{-2s}}{\varphi(N)} \sum_{\chi \bmod N} \left[\Big(\overline{\chi}(-v') + \overline{\chi}(v') \Big) \sum_{n=1}^{\infty} \frac{\varphi(n)\chi(n)}{n^{2s}} \right] \\ &= 2\sqrt{\pi}\, \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{N^{-2s}}{\varphi(N)} \sum_{\substack{\chi \bmod N \\ \text{even}}} \left[\overline{\chi}(v') \sum_{n=1}^{\infty} \frac{\varphi(n)\chi(n)}{n^{2s}} \right]. \end{aligned}$$

Since for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\sum_{n=1}^{\infty} \frac{\varphi(n)\chi(n)}{n^{2s}} = \prod_{p} \left(1 + \frac{\varphi(p)}{p^{2s}} + \frac{\varphi(p^2)}{p^{4s}} + \ldots \right)$$

$$= \prod_p \left(1 - \frac{\chi(p)}{p^{2s-1}}\right)^{-1} \prod_p \left(1 - \frac{\chi(p)}{p^{2s}}\right).$$

Finally, the Euler product of the Dirichlet L-functions (see Appendix D.2) implies the identity

$$\sum_{n=1}^{\infty} \frac{\varphi(n)\chi(n)}{n^{2s}} = \frac{L(2s-1,\chi)}{L(2s,\chi)},$$

and the result follows. □

Notation 2.2.10. Let $v \in (\mathbb{Z}/N\mathbb{Z})^\times$. We set the constant

$$\kappa_{N,v} := \sum_{\substack{\chi \neq \chi_0 \\ \text{even}}} \overline{\chi}(v') \frac{L(1,\chi)}{L(2,\chi)}, \tag{2.7}$$

where the sum runs over all non principal Dirichlet characters χ modulo N satisfying $\chi(-1) = 1$ and $v' \in \mathbb{Z}$ such that $vv' \equiv 1 \bmod N$.

Remark 2.2.11. The constant (2.7) is well-defined. Indeed, on the one hand the function $L(s,\chi)$ is holomorphic at $s = 1$ provided that $\chi \neq \chi_0$ (see [Apo76, Theorem 12.5 (c), p. 255]). On the other hand, we have $L(2,\chi) \neq 0$ since the inequalities

$$\frac{\zeta(2\mathrm{Re}(s))}{\zeta(\mathrm{Re}(s))} < |L(s,\chi)| \leq \zeta(\mathrm{Re}(s))$$

hold for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and χ any Dirichlet character modulo N.

Theorem 2.2.12. *Let $N \geq 3$ be an odd square-free integer and $0_v = [1 : v']$ with v, v' as in Notation 2.2.6. Then the following identities hold*

$$\mathscr{C}_{\infty\infty}^{\Gamma(N)} = 2v_{\Gamma(N)}^{-1}\left(\mathscr{C} - \sum_{p|N} \frac{p^2}{p^2-1}\log(p) - \log(N)\right),$$

$$\mathscr{C}_{0_v\infty}^{\Gamma(N)} = 2v_{\Gamma(N)}^{-1}\left(\mathscr{C} + \frac{1}{2}\sum_{p|N} \frac{1+2p-p^2}{p^2-1}\log(p) - \frac{1}{2}\log(N)\right) + \frac{2\pi}{N^2\varphi(N)}\kappa_{N,v},$$

where $\mathscr{C}$ resp. $\kappa_{N,v}$ is the constant given by (2.1) *and* (2.7)*, respectively.*

Proof. For the proof of the first identity, note that

$$\varphi_{\infty\infty}^{\Gamma(N)}(s) = \frac{v_{\Gamma(N)}^{-1}}{s-1} + 2v_{\Gamma(N)}^{-1}\left(\mathscr{C} - \sum_{p|N} \frac{p^2}{p^2-1}\log(p) - \log(N)\right) + O(s-1),$$

which is obtained from Proposition 2.2.9 using (2.2) and the Laurent expansion

(A.4) of Appendix A.

For the proof of the second identity, note that

$$\varphi_{0_v\infty}^{\Gamma(N)}(s) = 2G(s)\frac{N^{-2s}}{\varphi(N)}\left(\frac{L(2s-1,\chi_0)}{L(2s,\chi_0)} + \sum_{\substack{\chi\neq\chi_0\\ \text{even}}} \overline{\chi}(v')\frac{L(2s-1,\chi)}{L(2s,\chi)}\right) \tag{2.8}$$

by virtue of Proposition 2.2.9, where $G(s) = \sqrt{\pi}\,\Gamma(s-1/2)/\Gamma(s)$. The identity

$$\frac{L(2s-1,\chi_0)}{L(2s,\chi_0)} = N^s\frac{\zeta(2s-1)}{\zeta(2s)}\prod_{p|N}\frac{p^s-p^{1-s}}{p^{2s}-1},$$

which can be deduced from the Euler product of L-functions (see Appendix D.2), gives

$$2G(s)\frac{N^{-2s}}{\varphi(N)}\left(\frac{L(2s-1,\chi_0)}{L(2s,\chi_0)}\right) = 2G(s)\frac{N^{-s}}{\varphi(N)}\frac{\zeta(2s-1)}{\zeta(2s)}\prod_{p|N}\frac{p^s-p^{1-s}}{p^{2s}-1}.$$

Using (2.2), the Laurent expansions (A.3) and (A.5) of Appendix A, and the fact that N is square-free, we deduce that the right hand side of the previous identity has the following Laurent expansion at $s=1$,

$$\frac{v_{\Gamma(N)}^{-1}}{s-1} + v_{\Gamma(N)}^{-1}\left(2\mathscr{C} + \sum_{p|N}\frac{1+p-p^2}{p^2-1}\log(p) - \log(N)\right) + O(s-1). \tag{2.9}$$

Besides, since the second summand inside the brackets of (2.8) is holomorphic on $\mathbb{C}$, we have

$$2G(s)\frac{N^{-2s}}{\varphi(N)}\left(\sum_{\substack{\chi\neq\chi_0\\ \text{even}}} \overline{\chi}(v')\frac{L(2s-1,\chi)}{L(2s,\chi)}\right) = \frac{2\pi}{N^2\varphi(N)}\kappa_{N,v} + O(s-1). \tag{2.10}$$

Consequently, the expansions (2.9) and (2.10) provide the Laurent expansion of (2.8) at $s=1$; hence, the scattering constant $\mathscr{C}_{\infty 0_v}^{\Gamma(N)}$ is clear. $\square$

2.3 The cases $\Gamma_0(N)$ and $\Gamma_1(N)$

A subgroup $\Gamma\subset \mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$ for some $N\geq 1$ is called a *congruence subgroup*. In this section we focus on the following two congruence subgroups and study their interaction with $\Gamma(N)$.

Definition 2.3.1. The *congruence subgroups* $\Gamma_0(N)$ and $\Gamma_1(N)$ are defined by

$$\Gamma_0(N) := \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c\equiv 0 \bmod N\right\},$$

$$\Gamma_1(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}.$$

Remark 2.3.2. The subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ are indeed congruence subgroups since we have the inclusions $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$. Note that if $N = 1$, then we have $\Gamma(1) = \Gamma_1(1) = \Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$.

Lemma 2.3.3. *Let $N \geq 3$ be an integer. Then the following identities hold*

$$v_{\Gamma_1(N)} = \frac{\pi}{6} N \varphi(N) \prod_{p \mid N} \left(1 + \frac{1}{p}\right),$$

$$v_{\Gamma_0(N)} = \frac{\pi}{3} N \prod_{p \mid N} \left(1 + \frac{1}{p}\right).$$

Proof. For the proof we refer the reader to [Miy06, Theorem 4.2.5, p. 106] and Proposition 1.2.8. □

Lemma 2.3.4. *Let $N \geq 5$ be an integer. Then the following assertions hold:*

(a) The congruence subgroup $\Gamma_1(N)$ has no elliptic points, that is, we have $e_{\Gamma_1(N)} = 0$.

(b) The number of cusps of $\Gamma_1(N)\backslash\mathbb{H}$ is given by

$$c_{\Gamma_1(N)} = \frac{1}{2} \sum_{d \mid N} \varphi(d)\varphi(N/d).$$

(c) Suppose that N is square-free. Then the genus $g_{\Gamma_1(N)}$ of $X(\Gamma_1(N))$ is given by

$$g_{\Gamma_1(N)} = 1 + \frac{v_{\Gamma_1(N)}}{4\pi}\left(1 - \frac{6\, d(N)}{\sigma(N)}\right).$$

Proof. For the proof of parts (a) and (b) we refer the reader to [Miy06, Theorem 4.2.9, p. 111]. For part (c), note that if N is square-free, then we have $c_{\Gamma_1(N)} = (1/2)\varphi(N)d(N)$. In addition, it can be easily verified that

$$c_{\Gamma_1(N)} = \frac{3}{\pi}\frac{d(N)}{\sigma(N)} v_{\Gamma_1(N)},$$

where $\sigma(N)$ can be written as $\sigma(N) = \prod_{p \mid N}(p+1)$ since N is square-free. □

Lemma 2.3.5. *Let $N \geq 3$ be an integer. Then the following assertions hold:*

(a) *The congruence subgroup $\Gamma_0(N)$ may have elliptic points. More precisely, we have*

$$\nu_2(\Gamma_0(N)) = \begin{cases} 0, & 4 \mid N; \\ \prod_{p \mid N} \left(1 + \left(\frac{-1}{p}\right)\right), & 4 \nmid N; \end{cases}$$

$$\nu_3(\Gamma_0(N)) = \begin{cases} 0, & 9 \mid N; \\ \prod_{p \mid N} \left(1 + \left(\frac{-3}{p}\right)\right), & 9 \nmid N. \end{cases}$$

Here, $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

(b) *The number of cusps of $\Gamma_0(N)\backslash\mathbb{H}$ is*

$$c_{\Gamma_0(N)} = \sum_{d \mid N} \varphi\Big(\gcd(d, N/d)\Big).$$

(c) *Suppose that N is a square-free integer relatively prime to 6. Then the genus $g_{\Gamma_0(N)}$ of $X(\Gamma_0(N))$ is given by*

$$g_{\Gamma_0(N)} = 1 + \frac{1}{12}\prod_{p \mid N}(p+1) - \frac{1}{4}\prod_{p \mid N}\left(1 + \left(\frac{-1}{p}\right)\right) - \frac{1}{3}\prod_{p \mid N}\left(1 + \left(\frac{-3}{p}\right)\right) - \frac{1}{2}d(N).$$

Proof. For the proof we refer the reader to [Miy06, Theorem 4.2.7, p. 108] and [AU97, p. 60]. □

Theorem 2.3.6. *Let $\Gamma \supset \Gamma(N)$ be a congruence subgroup and $q_1, q_2 \in C_\Gamma$ two cusps having widths $w_{q_1}^\Gamma$ and $w_{q_2}^\Gamma$, respectively. Let $q_{1,1}, \ldots, q_{1,r} \in C_{\Gamma(N)}$ denote all the cusps of $\Gamma(N)\backslash\mathbb{H}$ which are Γ-equivalent to q_1. Suppose that $q_{2,1} \in C_{\Gamma(N)}$ is a cusp of $\Gamma(N)\backslash\mathbb{H}$ that is Γ-equivalent to q_2. Then the following identity holds*

$$\mathscr{C}_{q_1 q_2}^\Gamma = v_\Gamma^{-1} \log\left(\frac{N^2}{w_{q_1}^\Gamma w_{q_2}^\Gamma}\right) + \frac{N}{w_{q_1}^\Gamma}\sum_{j=1}^{r} \mathscr{C}_{q_{1,j} q_{2,1}}^{\Gamma(N)},$$

where r is the positive integer uniquely determined by the equality

$$r = (N v_\Gamma)^{-1} w_{q_1}^\Gamma v_{\Gamma(N)}.$$

Proof. For the proof, let σ_{q_1}, $\sigma_{q_{1,j}}$, and $\sigma_{q_{2,1}}$ be scaling matrices of q_1, $q_{1,j}$ $(j = 1, \ldots, r)$, and $q_{2,1}$, respectively. We choose

$$\sigma_{q_2} \coloneqq \gamma \sigma_{q_{2,1}} a(N^{-1/2}) a\Big((w_{q_2}^\Gamma)^{1/2}\Big), \tag{2.11}$$

where $\gamma \in \Gamma$ is such that $\gamma q_{2,1} = q_2$. We claim that σ_{q_2} is a scaling matrix of q_2. Indeed, to see the first condition of Definition 1.2.12, note that

$$\begin{aligned}\sigma_{q_2}\infty &= \gamma\sigma_{q_{2,1}} a(N^{-1/2}) a\Big((w^{\Gamma}_{q_2})^{1/2}\Big)\infty \\ &= \gamma\sigma_{q_{2,1}}\infty = \gamma q_{2,1} = q_2.\end{aligned}$$

For the other condition, we have the following identities

$$\begin{aligned}\{\pm I\}\cdot\sigma_{q_2}^{-1}\Gamma_{q_2}\sigma_{q_2} &= \{\pm I\}\cdot a\Big((N/w^{\Gamma}_{q_2})^{1/2}\Big)\sigma_{q_{2,1}}^{-1}\gamma^{-1}\Gamma_{q_2}\gamma\sigma_{q_{2,1}} a\Big((N/w^{\Gamma}_{q_2})^{-1/2}\Big) \\ &= \{\pm I\}\cdot a\Big((w^{\Gamma}_{q_2})^{-1/2})\Big) g_{q_{2,1}}^{-1}\Gamma_{q_{2,1}} g_{q_{2,1}} a\Big((w^{\Gamma}_{q_2})^{1/2}\Big) \\ &= \{\pm I\}\cdot a\Big((w^{\Gamma}_{q_2})^{-1/2})\Big) B(w^{\Gamma}_{q_2}) a\Big((w^{\Gamma}_{q_2})^{1/2})\Big) = \{\pm I\}\cdot B;\end{aligned}$$

here, for the second equality we used (2.4) and the identity $\gamma^{-1}\Gamma_{q_2}\gamma = \Gamma_{q_{2,1}}$ since $\gamma q_{2,1} = q_2$, and for the third equality we used $g_{q_{2,1}}^{-1}\Gamma_{q_{2,1}} g_{q_{2,1}} = B(w^{\Gamma}_{q_{2,1}})$ and $w^{\Gamma}_{q_{2,1}} = w^{\Gamma}_{q_2}$.

Now, by [Pos07, Satz 5.21, p. 75], we know that

$$E^{\Gamma}_{q_1}(w,s) = \left(\frac{N}{w^{\Gamma}_{q_1}}\right)^{s} \sum_{j=1}^{r} E^{\Gamma(N)}_{q_{1,j}}(w,s). \tag{2.12}$$

Taking $w = \sigma_{q_{2,1}} z$ with $z = x+iy$, we have on the left hand side of (2.12) the following identities

$$\begin{aligned}&E^{\Gamma}_{q_1}(\sigma_{q_{2,1}} z, s) = E^{\Gamma}_{q_1}\left(\gamma^{-1}\sigma_{q_2}\left(\frac{N}{w^{\Gamma}_{q_2}} z\right), s\right) \\ &= \delta_{q_1 q_2}\left(\frac{Ny}{w^{\Gamma}_{q_2}}\right)^{s} + \varphi^{\Gamma}_{q_1 q_2}(s)\left(\frac{Ny}{w^{\Gamma}_{q_2}}\right)^{1-s} + \sum_{m\neq 0} c_m(q_2;0) W_{0,ir}\left(4\pi|m|\frac{Nmy}{w^{\Gamma}_{2}}\right) e^{2\pi i\frac{Nmx}{w^{\Gamma}_{q_2}}};\end{aligned} \tag{2.13}$$

here, in the first equality we used (2.11) and in the second equality, the Fourier expansion of the Eisenstein series of weight 0 given in Section 1.4. On the right hand side of (2.12), we have

$$\begin{aligned}\left(\frac{N}{w^{\Gamma}_{q_1}}\right)^{s} \sum_{j=1}^{r} E^{\Gamma(N)}_{q_{1,j}}(\sigma_{q_{2,1}} z, s) = \left(\frac{N}{w^{\Gamma}_{q_1}}\right)^{s} \sum_{j=1}^{r} \Bigg\{&\delta_{q_{1,j} q_{2,1}} y^{s} + \varphi^{\Gamma(N)}_{q_{1,j} q_{2,1}}(s) y^{1-s} \\ &+ \sum_{n\neq 0} c'_m(q_{2,1};0) W_{0,ir}(4\pi|n|y) e^{2\pi i n x}\Bigg\}.\end{aligned} \tag{2.14}$$

Next, let us consider the constant terms

$$L_y(s) := \delta_{q_1 q_2}\left(\frac{Ny}{w^{\Gamma}_{q_2}}\right)^{s} + \varphi^{\Gamma}_{q_1 q_2}(s)\left(\frac{Ny}{w^{\Gamma}_{q_2}}\right)^{1-s},$$

$$R_y(s) := \left(\frac{N}{w_{q_1}^{\Gamma}}\right)^s \sum_{j=1}^{r} \left\{\delta_{q_{1,j}q_{2,1}} y^s + \varphi_{q_{1,j}q_{2,1}}^{\Gamma(N)}(s) y^{1-s}\right\},$$

of (2.13) and (2.14), respectively. In order to compare the Laurent expansions of (2.13) and (2.14) at $s = 1$, we choose y sufficiently large in such a way that

$$E_{q_1}^{\Gamma}(\sigma_{q_{2,1}} z, s) - L_y(s) \quad \text{and} \quad \left(\frac{N}{w_{q_1}^{\Gamma}}\right)^s \sum_{j=1}^{r} E_{q_{1,j}}^{\Gamma(N)}(\sigma_{q_{2,1}} z, s) - R_y(s)$$

do not have contributions in our analysis. This can be done because the previous expressions tend to zero faster than $L_y(s)$ and $R_y(s)$ do, as $y \to \infty$ (see Remark 1.4.7).
The Laurent expansions of $L_y(s)$ and $R_y(s)$ at $s = 1$ are given by

$$L_y(s) = \frac{v_{\Gamma}^{-1}}{s-1} + \left[\mathscr{C}_{q_1 q_2}^{\Gamma} - v_{\Gamma}^{-1} \log\left(\frac{N}{w_{q_2}^{\Gamma}}\right) + \frac{\delta_{q_1 q_2} N}{w_{q_2}^{\Gamma}} y - v_{\Gamma}^{-1} \log(y)\right] + O_y(s-1),$$

$$R_y(s) = \frac{\frac{Nr}{w_{q_1}^{\Gamma}} v_{\Gamma(N)}^{-1}}{s-1} + \frac{N}{w_{q_1}^{\Gamma}}\left[y \sum_{j=1}^{r} \delta_{q_{1,j}q_{2,1}} + \sum_{j=1}^{r} \mathscr{C}_{q_{1,j}q_{2,1}}^{\Gamma(N)} + r v_{\Gamma(N)}^{-1} \log\left(\frac{N}{y w_{q_1}^{\Gamma}}\right)\right] + O_y(s-1).$$

Observe that by comparing the residues of $L_y(s)$ and $R_y(s)$ at $s = 1$, we obtain

$$r = \frac{w_{q_1}^{\Gamma}}{N} \frac{v_{\Gamma(N)}}{v_{\Gamma}}.$$

Now, if we compare the constant terms of $L_y(s)$ and $R_y(s)$, then after cancelling the $\log(y)$ term, we have

$$\mathscr{C}_{q_1 q_2}^{\Gamma} - v_{\Gamma}^{-1} \log\left(\frac{N}{w_{q_2}^{\Gamma}}\right) + \frac{\delta_{q_1 q_2} N}{w_{q_2}^{\Gamma}} y = \frac{N}{w_{q_1}^{\Gamma}}\left[y \sum_{j=1}^{r} \delta_{q_{1,j}q_{2,1}} + \sum_{j=1}^{r} \mathscr{C}_{q_{1,j}q_{2,1}}^{\Gamma(N)} + r v_{\Gamma(N)}^{-1} \log\left(\frac{N}{w_{q_1}^{\Gamma}}\right)\right].$$

Rearranging the previous identity, we have

$$\mathscr{C}_{q_1 q_2}^{\Gamma} - v_{\Gamma}^{-1} \log\left(\frac{N^2}{w_{q_1}^{\Gamma} w_{q_2}^{\Gamma}}\right) - \frac{N}{w_{q_1}^{\Gamma}} \sum_{j=1}^{r} \mathscr{C}_{q_{1,j}q_{2,1}}^{\Gamma(N)} + y\left(\frac{\delta_{q_1 q_2} N}{w_{q_2}^{\Gamma}} - \frac{N}{w_{q_1}^{\Gamma}} \sum_{j=1}^{r} \delta_{q_{1,j}q_{2,1}}\right) = 0.$$

Considering now y as a variable, we finally obtain the identities

$$\mathscr{C}_{q_1 q_2}^{\Gamma} = v_{\Gamma}^{-1} \log\left(\frac{N^2}{w_{q_1}^{\Gamma} w_{q_2}^{\Gamma}}\right) + \frac{N}{w_{q_1}^{\Gamma}} \sum_{j=1}^{r} \mathscr{C}_{q_{1,j}q_{2,1}}^{\Gamma(N)}$$

$$\delta_{q_1 q_2} = \frac{w_{q_2}^{\Gamma}}{w_{q_1}^{\Gamma}} \sum_{j=1}^{r} \delta_{q_{1,j}q_{2,1}}.$$

This concludes the proof. $\square$

Finally, by using Theorem 2.3.6, we deduce the desired scattering constants for the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$.

Corollary 2.3.7. *Let $N \geq 3$ be an odd square-free integer and $0_v = [1 : v']$ with v, v' as in Notation 2.2.6. Then the following identities hold*

$$\mathscr{C}_{\infty\infty}^{\Gamma_1(N)} = 2v_{\Gamma_1(N)}^{-1}\left(\mathscr{C} - \sum_{p|N} \frac{p^2}{p^2-1}\log(p)\right),$$

$$\mathscr{C}_{\infty 0_v}^{\Gamma_1(N)} = 2v_{\Gamma_1(N)}^{-1}\left(\mathscr{C} + \frac{1}{2}\sum_{p|N} \frac{1+2p-p^2}{p^2-1}\log(p)\right) + \frac{2\pi}{N\varphi(N)}\kappa_{N,v},$$

where $\kappa_{N,v}$ is the constant given by (2.7).

Proof. For the proof of the first identity, we choose $q_1 = q_2 = q_{2,1} = \infty$. Since $w_\infty^{\Gamma_1(N)} = 1$ and $r = 1$, we have $q_{1,1} = \infty$ and by Theorem 2.3.6, we obtain

$$\mathscr{C}_{\infty\infty}^{\Gamma_1(N)} = 2v_{\Gamma_1(N)}^{-1}\log(N) + N\mathscr{C}_{\infty\infty}^{\Gamma(N)}.$$

The result follows by the first identity of Theorem 2.2.12.

For the proof of the second identity, we choose $q_1 = \infty$ and $q_2 = q_{2,1} = 0_v$. Since $w_{q_1}^{\Gamma_1(N)} = 1$, $w_{q_2}^{\Gamma_1(N)} = N$ and $r = 1$, we have $q_{1,1} = \infty$ and by Theorem 2.3.6, we obtain

$$\mathscr{C}_{\infty 0_v}^{\Gamma_1(N)} = \log(N)v_{\Gamma_1(N)}^{-1} + N\mathscr{C}_{\infty 0_v}^{\Gamma(N)}.$$

Since $\varphi_{0_v\infty}^{\Gamma(N)}(s) = \varphi_{\infty 0_v}^{\Gamma(N)}(s)$, the result follows by the second identity of Theorem 2.2.12. $\square$

Corollary 2.3.8. *Let $N \geq 3$ be an odd square-free integer. Then the following identities hold*

$$\mathscr{C}_{\infty\infty}^{\Gamma_0(N)} = 2v_{\Gamma_0(N)}^{-1}\left(\mathscr{C} - \sum_{p|N} \frac{p^2}{p^2-1}\log(p)\right),$$
$$\mathscr{C}_{\infty 0}^{\Gamma_0(N)} = 2v_{\Gamma_0(N)}^{-1}\left(\mathscr{C} + \frac{1}{2}\sum_{p|N} \frac{1+2p-p^2}{p^2-1}\log(p)\right).$$

Proof. For the proof of the first identity, we choose $q_1 = q_2 = q_{2,1} = \infty$ in Theorem 2.3.6; therefore, we have $w_{q_1}^{\Gamma_0(N)} = w_{q_2}^{\Gamma_0(N)} = 1$ and $r = \varphi(N)/2$. It remains to determine the scattering constant $\mathscr{C}_{\infty_l\infty}^{\Gamma(N)}$, where we are using ∞_l to denote $q_{1,l}$.

From [Hej83, Theorem 5.12, p. 549], we have

$$\begin{aligned}\varphi^{\Gamma(N)}_{\infty_l\infty}(s) &= 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}N^{1-2s}\prod_{p|N}\frac{1}{p^{2s}-1} \\ &= 2\varphi^{\mathrm{SL}_2(\mathbb{Z})}_{\infty\infty}(s)N^{1-2s}\prod_{p|N}\frac{1}{p^{2s}-1}\,;\end{aligned}$$

here, the cusp ∞ resp. ∞_l is represented, in Hejhal's notation (see p. 541), by the triple $(j,x,A)=(0,1,1)$ and $(j,x,A)=(0,l,1)$, respectively. Now, using (2.2) and the Laurent expansion (A.4) of Appendix A, we obtain

$$\varphi^{\Gamma(N)}_{\infty_l\infty}(s)=\frac{v^{-1}_{\Gamma(N)}}{s-1}+2v^{-1}_{\Gamma(N)}\left(\mathscr{C}-\sum_{p|N}\frac{p^2}{p^2-1}\log(p)-\log(N)\right)+O(s-1),$$

which in turn gives the scattering constant

$$\mathscr{C}^{\Gamma(N)}_{\infty_l\infty}=2v^{-1}_{\Gamma(N)}\left(\mathscr{C}-\sum_{p|N}\frac{p^2}{p^2-1}\log(p)-\log(N)\right).$$

Thus the result follows.

For the second identity, we choose $q_1=\infty$ and $q_2=q_{2,1}=0$ in Theorem 2.3.6; therefore, we have $w^{\Gamma_0(N)}_{q_1}=1$, $w^{\Gamma_0(N)}_{q_2}=N$, and $r=\varphi(N)/2$. It remains to determine the scattering constant $\mathscr{C}^{\Gamma(N)}_{\infty_l0}$.

From [Hej83, Theorem 5.12, p. 549], we have

$$\varphi^{\Gamma(N)}_{\infty_l0}(s)=2\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{N^{-2s}}{\varphi(N)}\sum_{\substack{\chi \bmod N\\ \text{even}}}\overline{\chi}(v')\frac{L(2s-1,\chi)}{L(2s,\chi)}\,;$$

here, the cusp 0 is represented by $(j,x,A)=(N-1,1,N)$ in Hejhal's notation. Consequently, we obtain the following Laurent expansion at $s=1$

$$\varphi^{\Gamma(N)}_{\infty_l0}(s)=\frac{v^{-1}_{\Gamma(N)}}{s-1}+\mathscr{C}^{\Gamma(N)}_{\infty_l0}+O(s-1),$$

where

$$\mathscr{C}^{\Gamma(N)}_{\infty_l0}=2v^{-1}_{\Gamma(N)}\left(\mathscr{C}+\frac{1}{2}\sum_{p|N}\frac{1+2p-p^2}{p^2-1}\log(p)-\frac{1}{2}\log(N)\right)+\frac{2\pi}{N^2\varphi(N)}\kappa_{N,v}.$$

This concludes the proof. □

Remark 2.3.9. Note that the method we have used to determine the scattering constants of the subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ differs from what can be found in the literature. We point out that our results for $\mathscr{C}^{\Gamma_0(N)}_{\infty\infty}$ and $\mathscr{C}^{\Gamma_0(N)}_{\infty0}$ coincide with [AU97, p. 59] and [AU97, p. 67], respectively.

Remark 2.3.10. Note that there is a mistake in Lemma 3.6 of [May14, p. 123] when labeling the cusps ∞_d. Given an embedding $\sigma : \mathbb{Q}(\zeta_N) \longrightarrow \mathbb{C}$, it can be proved that $\infty^\sigma = [0 : 1]$ and $0^\sigma = [1 : d]$ hold for some $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. This can be proved, e.g., using the same strategy given in the proof of Proposition 5.1.6. With this, the scattering function $\varphi_{0\infty_d}$ in [May14] actually depends on d. Despite this mistake, the asymptotic expansion of [May14] for the self-intersection number is not affected.

Chapter 3

Arakelov theory on modular curves

In this chapter, we describe the minimal regular models of the modular curves associated to the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$. Moreover, we obtain explicit expressions for the analytic and geometric contributions of the self-intersection of the relative dualizing sheaf in each case.

In Section 3.1, we introduce the notion of an arithmetic surface and review the construction of the relative dualizing sheaf. In particular, we specify the pullbacks of the latter to the fibers at infinity.

In Section 3.2, we review the Arakelov intersection theory on arithmetic surfaces.

In Section 3.3, we define the moduli problems of elliptic curves which will provide the minimal regular models of the aforementioned modular curves. In view of our guiding philosophy, we first focus on the moduli problem $[\Gamma(N)]$ of $\Gamma(N)$-structures on elliptic curves and then define the other moduli problems as suitable quotients of $[\Gamma(N)]$ by subgroups of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

In Section 3.4, we use the theorems of Manin–Drinfeld, Falting–Hriljac, and Néron–Tate to obtain a formula for the self-intersection of the relative dualizing sheaf.

The main references for this part are [Ara74], [KM85], [Liu02], and [Mor14].

3.1 Arithmetic surfaces

For the sequel, we let K be a number field with ring of integers $\mathcal{O}_K$ and set $S \coloneqq \mathrm{Spec}(\mathcal{O}_K)$.

Let $\mathcal{X}/S$ be an S-scheme. For $s \in S$, we set $\mathcal{X}_s \coloneqq \mathcal{X} \times_S \mathrm{Spec}(k(s))$, where $k(s)$ denotes the residue field at s. If $s \in S$ is a closed point, then $\mathcal{X}_s$ is a *closed* *fiber*, whereas if η is the generic point of S, then $\mathcal{X}_\eta$ is called the *generic fiber of* $\mathcal{X}$.

Definition 3.1.1. An *arithmetic surface* over S is a regular 2-dimensional integral scheme $\mathcal{X}$ endowed with a projective and flat structural morphism such that the generic fiber $\mathcal{X}_\eta$ is a geometrically connected curve over K.

In the next proposition we gather some properties about the fibers of an arithmetic surface.

Proposition 3.1.2. *Let $\mathcal{X}/S$ be an arithmetic surface. Then the following assertions hold:*

(a) The generic fiber $\mathcal{X}_\eta$ is an integral curve over K.

(b) For each $s \in S$, the fiber $\mathcal{X}_s$ is a geometrically connected projective curve over $k(s)$.

(c) If $\mathcal{X}_\eta$ is a smooth curve, then $\mathcal{X}_s$ is smooth for all $s \in S$ except a finite number of s.

Proof. For the proof we refer the reader to [Liu02, Lemma 3.3, p. 348], [Liu02, Corollary 3.6, p. 350] and [Liu02, Proposition 3.11, p. 352]. □

Let $\mathcal{X}/S$ be an arithmetic surface. Suppose that $Z \subset \mathcal{X}$ is an irreducible closed subscheme of codimension one, i.e., a *prime Weil divisor*. Then either Z is an irreducible component of a closed fiber, or Z is the closure in $\mathcal{X}$ of a closed point of $\mathcal{X}_\eta$ (see [Liu02, Proposition 3.4 (b), p. 349]). In the first case, we say that Z is an *irreducible vertical divisor*, whereas in the second case we say that Z is an *irreducible horizontal divisor*.

Definition 3.1.3. Let $Z^1(\mathcal{X})$ denote the free abelian group generated by all prime Weil divisors. A *Weil divisor* D is an element of the group $Z^1(\mathcal{X})$. We say that D is *horizontal* resp. *vertical* if each component of D is an irreducible horizontal or irreducible vertical divisor, respectively.

Let $K(\mathcal{X})$ be the field of rational functions of $\mathcal{X}$. Given a non-zero rational function $f \in K(\mathcal{X})$, there exists a Weil divisor, denoted by (f), associated to f. Such divisors are called *principal Weil divisors* and the collection of all principal Weil divisors forms a subgroup of $Z^1(\mathcal{X})$. In the sequel we write $\mathrm{Cl}(\mathcal{X})$ to denote the quotient of $Z^1(\mathcal{X})$ by the principal Weil divisors.

Definition 3.1.4. An $\mathcal{O}_\mathcal{X}$-module $\mathcal{L}$ over $\mathcal{X}$ is called *invertible* if it is locally free of rank 1.

In the sequel we write $\mathrm{Pic}(\mathcal{X})$ to denote the group of isomorphism classes of invertible $\mathcal{O}_{\mathcal{X}}$-modules over $\mathcal{X}/S$. The group operation in $\mathrm{Pic}(\mathcal{X})$ is given by the tensor product of $\mathcal{O}_{\mathcal{X}}$-modules and the inverse is given by $\mathcal{F}^{-1} := \mathcal{F}^{\vee}$, where $\mathcal{F}^{\vee} := \mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{O}_{\mathcal{X}})$ (see [Liu02, Exercise 1.5, p. 172]).

Let $D \in Z^1(\mathcal{X})$ and suppose that f_j is a local representation of D on the open subset $U_j \subset \mathcal{X}$. We associate to D an invertible $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{O}_{\mathcal{X}}(D)$ as follows: for each j, put $\mathcal{O}_{\mathcal{X}}(D)|_{U_j} := f_j^{-1} \cdot \mathcal{O}_{\mathcal{X}}|_{U_j}$.

Proposition 3.1.5. *Let $\mathcal{X}/S$ be an arithmetic surface. Then the assignment*

$$D \longmapsto \mathcal{O}_{\mathcal{X}}(D) \tag{3.1}$$

induces an isomorphism between $\mathrm{Cl}(\mathcal{X})$ *and* $\mathrm{Pic}(\mathcal{X})$.

Proof. For the proof we refer the reader to [Liu02, Corollary 1.19, p. 257]. □

In what follows, we introduce the relative dualizing sheaf of $\mathcal{X}/S$ which will play an important role in this thesis. In order to define it, we need to review some additional notions. For further details of these notions, we refer the reader to chapters 5 and 6 of [Liu02].

First, let us construct the so-called *conormal sheaf.* Since the arithmetic surface $\mathcal{X}/S$ is projective, the structural morphism of $\mathcal{X}/S$ factors through a closed immersion $i : \mathcal{X} \longrightarrow \mathbb{P}^n_S$ for some positive integer n, followed by the natural morphism $h : \mathbb{P}^n_S \longrightarrow S$. Let $\mathcal{I}$ be the sheaf of ideals defining the closed immersion i. Then the *conormal sheaf* is the $\mathcal{O}_{\mathcal{X}}$-module given by

$$\mathcal{C}_{\mathcal{X}/\mathbb{P}^n_S} := i^*(\mathcal{I}/\mathcal{I}^2).$$

Remark 3.1.6. The conormal sheaf $\mathcal{C}_{\mathcal{X}/\mathbb{P}^n_S}$ is in fact a locally free $\mathcal{O}_{\mathcal{X}}$-module by virtue of [Liu02, Corollary 3.8, p. 229]. Indeed, since the structural morphism of $\mathcal{X}/S$ is a l.c.i., the morphism $i : \mathcal{X} \longrightarrow \mathbb{P}^n_S$ is a regular immersion (see [Liu02, Corollary 3.22, p. 233]).

Next, let us construct the sheaf of *relative differentials of degree 1.* Consider the diagonal morphism $\Delta_h : \mathbb{P}^n_S \longrightarrow \mathbb{P}^n_S \times_S \mathbb{P}^n_S$ induced by the morphism $h : \mathbb{P}^n_S \longrightarrow S$. Then Δ_h is a closed immersion. Let $\mathcal{J}$ be the sheaf of ideals defining $\Delta_h(\mathbb{P}^n_S)$. Then the $\mathcal{O}_{\mathbb{P}^n_S}$-module of *relative differentials of degree 1* is given by

$$\Omega^1_{\mathbb{P}^n_S/S} := \Delta_h^*(\mathcal{J}/\mathcal{J}^2).$$

Definition 3.1.7. The *canonical sheaf* over $\mathcal{X}/S$ is the invertible $\mathcal{O}_{\mathcal{X}}$-module given by

$$\omega_{\mathcal{X}/S} := \det(\mathcal{C}_{\mathcal{X}/\mathbb{P}^n_S})^\vee \otimes_{\mathcal{O}_\mathcal{X}} \det(i^*(\Omega^1_{\mathbb{P}^n_S/S})),$$

where $i : \mathcal{X} \longrightarrow \mathbb{P}^n_S$ is a closed immersion.

Remark 3.1.8. The previous definition is independent of the choice of the projective embedding $i : \mathcal{X} \longrightarrow \mathbb{P}^n_S$. Furthermore, the canonical sheaf $\omega_{\mathcal{X}/S}$ is the relative dualizing sheaf in the sense of Grothendieck duality (see [Liu02, Theorem 4.32, p. 247]). For this reason we will from now on refer to $\omega_{\mathcal{X}/S}$ as the *relative dualizing sheaf of* $\mathcal{X}/S$.

In the next section we will consider the curve $X_\sigma/\mathbb{C}$ given by the base change

$$X_\sigma := \mathcal{X}_\eta \times_{K,\sigma} \operatorname{Spec}(\mathbb{C}) \tag{3.2}$$

with $\sigma : K \hookrightarrow \mathbb{C}$ an embedding of the number field K into $\mathbb{C}$. It turns out that X_σ defines a proper smooth curve over $\mathbb{C}$ via the second projection map; therefore, the analytification X_σ^{an} is a compact connected Riemann surface of genus g_σ which we always assume to be positive.

Proposition 3.1.9. *Let $\mathcal{X}/S$ be an arithmetic surface and $\sigma : K \hookrightarrow \mathbb{C}$ an embedding of the number field K into $\mathbb{C}$. Then the identity*

$$(\omega_{\mathcal{X}/S})|_{X_\sigma} \simeq \Omega^1_{X_\sigma}$$

holds, where X_σ is given by (3.2) *and $\Omega^1_{X_\sigma}$ denotes the space of global holomorphic* 1*-forms on X_σ.*

Proof. For the proof we refer the reader to [Liu02, Theorem 4.9 (b), p. 239]. □

3.2 Arakelov intersection theory

For the following considerations, we let $\mathcal{X}/S$ be an arithmetic surface and let $K(\mathbb{C})$ denote the set of all embeddings of the number field K into $\mathbb{C}$.

Suppose that X is a compact connected Riemann surface of genus $g_X \geq 1$. Recall that on X we have the canonical (1,1)-form μ_{can} and the operators d and d^c (see Section 1.6). Let $(L, ||\cdot||)$ be a pair consisting of a holomorphic line bundle L over X and a smooth hermitian metric $||\cdot||$ defined on L.

Definition 3.2.1. The *curvature* of $(L, ||\cdot||)$ is the $(1,1)$-form given by

$$c_1(L, ||\cdot||) := -\mathrm{d}\mathrm{d}^c \log ||s||^2,$$

where s is a non-zero rational section of L.

Definition 3.2.2. A metric $||\cdot||$ on L is called *admissible* if the identity

$$c_1(L, ||\cdot||) = \deg(L)\, \mu_{\mathrm{can}}$$

holds.

Lemma 3.2.3. *Let L be a line bundle over a compact connected Riemann surface. Then L admits an admissible metric which is uniquely determined up to multiplication by a positive constant.*

Proof. For the proof we refer the reader to [Mor14, Corollary 4.11, p. 98]. □

Remark 3.2.4. Let $P \in X$ and s the non-zero rational section which is the image of $1 \in \mathcal{O}_X$ under the map $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(P)$. Consider the metric $||\cdot||$ on $\mathcal{O}_X(P)$ defined by $-\log ||s||^2(Q) = g_{\mathrm{can}}(P,Q)$. Then condition (i) of Definition 1.6.3 implies that $||\cdot||$ is admissible, whereas condition (ii) determines $||\cdot||$ uniquely. Using this construction, we conclude that the line bundles $\mathcal{O}_X(D)$, with D a divisor of X, are canonically endowed with an admissible metric.

Let us go back to the situation of an arithmetic surface $\mathcal{X}/S$. Let $\sigma \in K(\mathbb{C})$ and X_σ the curve given by (3.2). Suppose that $\mathcal{L}$ is an invertible $\mathcal{O}_{\mathcal{X}}$-module over $\mathcal{X}/S$. We denote by L_σ the line bundle over X_σ^{an} corresponding to the pull-back $\mathcal{L}|_{X_\sigma}$.

Definition 3.2.5. A *metrized line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, \{||\cdot||_\sigma\}_{\sigma\in K(\mathbb{C})})$ over $\mathcal{X}/S$ is a pair consisting of an invertible $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{L}$ and a smooth hermitian metric $||\cdot||_\sigma$ on L_σ, for each $\sigma \in K(\mathbb{C})$.

Definition 3.2.6. A metrized line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \{||\cdot||_\sigma\}_{\sigma\in K(\mathbb{C})})$ over $\mathcal{X}/S$ is *admissible* if the pair $(L_\sigma, ||\cdot||_\sigma)$ is admissible, for all $\sigma \in K(\mathbb{C})$.

Now we proceed to define admissible metrics $||\cdot||_{\mathrm{can},\sigma}$ on $\Omega^1_{X_\sigma^{\mathrm{an}}}$ such that

$$\overline{\omega}_{\mathcal{X}/S} \coloneqq (\omega_{\mathcal{X}/S}, \{||\cdot||_{\mathrm{can},\sigma}\}_{\sigma\in K(\mathbb{C})})$$

becomes an admissible metrized line bundle. Given $\sigma \in K(\mathbb{C})$, we let $\Delta_\sigma : X_\sigma^{\mathrm{an}} \longrightarrow X_\sigma^{\mathrm{an}} \times X_\sigma^{\mathrm{an}}$ be the diagonal map and let Δ be the diagonal divisor on $X_\sigma^{\mathrm{an}} \times X_\sigma^{\mathrm{an}}$. We have an isomorphism

$$\alpha : \Delta_\sigma^* \mathcal{O}_{X_\sigma^{\mathrm{an}} \times X_\sigma^{\mathrm{an}}}(-\Delta) \;\simeq\; \Omega^1_{X_\sigma^{\mathrm{an}}},$$

called the *adjunction isomorphism*, where $\mathcal{O}_{X_\sigma^{\mathrm{an}} \times X_\sigma^{\mathrm{an}}}(-\Delta)$ denotes the sheaf of holomorphic functions vanishing on the diagonal. We define a metric $||\cdot||$ on $\mathcal{O}_{X_\sigma^{\mathrm{an}}\times X_\sigma^{\mathrm{an}}}(-\Delta)$ as follows: outside the diagonal, we put

$$-\log ||1||^2(P,Q) = g^{\sigma}_{\mathrm{can}}(P,Q),$$

where $g^{\sigma}_{\mathrm{can}}$ denotes the Green's function associated to $\mu_{\mathrm{can},\sigma}$. We extend this by continuity. Then we define the metric $||\cdot||_{\mathrm{can},\sigma}$ on $\Omega^1_{X_\sigma^{\mathrm{an}}}$ by requiring that the isomorphism α be an isometry.

Notation 3.2.7. In the sequel, we denote by $\overline{\omega}_{\mathcal{X}/S}$ the admissible metrized line bundle consisting of the relative dualizing sheaf $\omega_{\mathcal{X}/S}$ together with the metrics $||\cdot||_{\mathrm{can},\sigma}$ on $\Omega^1_{X_\sigma^{\mathrm{an}}}$.

Two admissible metrized line bundles $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ over $\mathcal{X}/S$ are *isomorphic* if there exists an isomorphism of invertible $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{L}\xrightarrow{\sim}\mathcal{M}$ such that it induces isometries between L_σ and M_σ, for all $\sigma\in K(\mathbb{C})$. In the sequel we will write $\widehat{\mathrm{Pic}}(\mathcal{X})$ to denote the set of all isomorphism classes of admissible metrized line bundles.

Let $\overline{\mathcal{L}}=(\mathcal{L},\{||\cdot||_{l,\sigma}\}_{\sigma\in K(\mathbb{C})})$ and $\overline{\mathcal{M}}=(\mathcal{M},\{||\cdot||_{m,\sigma}\}_{\sigma\in K(\mathbb{C})})$ be two admissible metrized line bundles over $\mathcal{X}/S$. The *tensor product* $\overline{\mathcal{L}}\otimes\overline{\mathcal{M}}$ is the metrized line bundle given by $(\mathcal{L}\otimes\mathcal{M},\{||\cdot||_\sigma\}_{\sigma\in K(\mathbb{C})})$, where $||\cdot||_\sigma$ denotes the product of $||\cdot||_{l,\sigma}$ and $||\cdot||_{m,\sigma}$. It can be verified that $\widehat{\mathrm{Pic}}(\mathcal{X})$ forms a group under the tensor product, where the inverse of $\overline{\mathcal{L}}$ is given by $\overline{\mathcal{L}}^{-1}:=(\mathcal{L}^\vee,\{||\cdot||^{-1}_{l,\sigma}\}_{\sigma\in K(\mathbb{C})})$.

Now, we introduce the so-called *Arakelov divisors* which provide a geometrical view of the metrized line bundles over $\mathcal{X}/S$.

Definition 3.2.8. An *Arakelov divisor* of $\mathcal{X}/S$ is a finite formal linear combination

$$\widehat{D}=D+\sum_{\sigma\in K(\mathbb{C})}\lambda_\sigma\cdot X_\sigma,$$

where $D\in Z^1(\mathcal{X})$, the λ_σ are real numbers, and X_σ as in (3.2).

Definition 3.2.9. A *principal Arakelov divisor* is an Arakelov divisor of the form

$$\widehat{(f)}=(f)+\sum_{\sigma\in K(\mathbb{C})}v_\sigma(f)\cdot X_\sigma,$$

where $f \in K(\mathcal{X})$ and

$$v_\sigma(f) := -\int_{X_\sigma^{\mathrm{an}}} \log|f \otimes_\sigma 1| \cdot \mu_{\mathrm{can},\sigma}$$

with $|\cdot|$ the absolute value on $\mathbb{C}$.

Two Arakelov divisors are *linearly equivalent* if their difference is a principal Arakelov divisor. In the sequel we will write $\widehat{\mathrm{Cl}}(\mathcal{X})$ to denote the set of all Arakelov divisors modulo linear equivalence.

Proposition 3.2.10. *Let $\mathcal{X}/S$ be an arithmetic surface. Then the assignment*

$$\widehat{D} = D + \sum_{\sigma \in K(\mathbb{C})} \lambda_\sigma \cdot X_\sigma \longmapsto \overline{\mathcal{O}_{\mathcal{X}}(D)} = \Big(\mathcal{O}_{\mathcal{X}}(D), \{\exp(-2\lambda_\sigma)||\cdot||_\sigma\}_{\sigma \in K(\mathbb{C})}\Big),$$

where $\mathcal{O}_{\mathcal{X}}(D)$ is the invertible $\mathcal{O}_{\mathcal{X}}$-module given by (3.1) *and $||\cdot||_\sigma$ denotes the canonical metric on $\mathcal{O}_{\mathcal{X}}(D)|_{X_\sigma}$ given by Remark 3.2.4, defines an isomorphism of the groups $\widehat{\mathrm{Cl}}(\mathcal{X})$ and $\widehat{\mathrm{Pic}}(\mathcal{X})$.*

Proof. For the proof we refer the reader to [Mor14, Lemma 4.15, p. 103]. □

Finally, we proceed to define the intersection pairing between two Arakelov divisors. For the moment we denote the structural morphism of $\mathcal{X}/S$ by $\xi : \mathcal{X} \longrightarrow S$.

Definition 3.2.11. Let $\widehat{D}$ and $\widehat{E}$ be two distinct irreducible Arakelov divisors, i.e., either a prime Weil divisor or X_σ for some $\sigma \in K(\mathbb{C})$. The *intersection number* $(\widehat{D}, \widehat{E})_{\mathrm{Ar}}$ is defined as follows

(I) If $\widehat{E} = X_\sigma$, then we define

$$(\widehat{D}, X_\sigma)_{\mathrm{Ar}} := \begin{cases} 0, & \text{if } D \text{ is a component of a closed fiber;} \\ 0, & \text{if } D = X_{\sigma'} \text{ with } \sigma' \in K(\mathbb{C}); \\ m, & \text{if } D \text{ is horizontal of degree } m \text{ on } \mathcal{X}_\eta. \end{cases}$$

(II) If $\widehat{D} = D$ and $\widehat{E} = E$ are prime Weil divisors, then we define

$$(D, E)_{\mathrm{Ar}} := (D, E)_{\mathrm{fin}} + (D, E)_\infty,$$

where $(D, E)_{\mathrm{fin}}$ denotes the usual intersection number on arithmetic surfaces, namely, we have

$$(D, E)_{\mathrm{fin}} := \sum_{\substack{\mathfrak{p} \in S \\ \mathfrak{p} \text{ closed}}} \Bigg(\sum_{\substack{x \in \mathcal{X} \\ \xi(x) = \mathfrak{p}}} \mathrm{length}_{\mathcal{O}_{\mathcal{X},x}} \Big(\mathcal{O}_{\mathcal{X},x}/\langle f_x, g_x\rangle\Big)\Bigg) \log(\# k(\mathfrak{p})),$$

where f_x and g_x are local equations of D and E, respectively; whereas

$$(D,E)_\infty \coloneqq \sum_{\sigma\in K(\mathbb{C})} (D,E)_\sigma;$$

here, $(D,E)_\sigma$ is defined as follows:

(i) If either D or E is a component of a vertical divisor, then we define $(D,E)_\sigma \coloneqq 0$.

(ii) If both are irreducible horizontal, then D (resp. E) is the Zariski closure of an L-rational point P (resp. F-rational point Q) of $\mathcal{X}_\eta$ with L (resp. F) an extension of $\mathbb{Q}$. Let Σ_L (resp. Σ_F) denotes the set of embeddings of L (resp. F) into $\mathbb{C}$ extending σ. Then we define

$$(D,E)_\sigma \coloneqq \sum_{(\alpha,\beta)\in\Sigma_L\times\Sigma_F} g^\sigma_{\mathrm{can}}(P^\alpha, Q^\beta).$$

Remark 3.2.12. We define the self-intersection of an Arakelov divisor via the adjunction formula (see, e.g., [Mor14, Lemma 4.26, p. 113]).

Proposition 3.2.13. *Let $\mathcal{X}/S$ be an arithmetic surface. Then the intersection number $(\widehat{D},\widehat{E})_{\mathrm{Ar}}$ given by Definition 3.2.11 induces a symmetric bilinear pairing on $\widehat{\mathrm{Cl}}(\mathcal{X})$.*

Proof. For the proof we refer the reader to [Ara74]. □

Remark 3.2.14. Using the isomorphism of Proposition 3.2.10, we can talk about the intersection of two admissible metrized line bundles.

Notation 3.2.15. In the sequel, we write $\overline{\omega}^2_{\mathcal{X}/S}$ to denote the self-intersection number $(\overline{\omega}_{\mathcal{X}/S}, \overline{\omega}_{\mathcal{X}/S})_{\mathrm{Ar}}$ and we call it the *self-intersection of the relative dualizing sheaf.*

3.3 Minimal regular models of modular curves for congruence subgroups

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and X_Γ/K the proper smooth algebraic curve associated to $X(\Gamma)$; here, K is a suitable number field. In the following, we will show the existence of a minimal regular model of X_Γ/K, i.e., an arithmetic surface $\mathcal{X}_\Gamma/\mathrm{Spec}(\mathcal{O}_K)$ such that $(\mathcal{X}_\Gamma)_\eta \simeq X_\Gamma$. We restrict ourselves to the congruence subgroups $\Gamma = \Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$.

Definition 3.3.1. Let T be an arbitrary scheme. An *elliptic curve over* T is a proper smooth T-scheme E/T with relative dimension one and geometrically connected fibers of genus one, endowed with a section in $E(T)$.

Lemma 3.3.2. *Let E/T be an elliptic curve and let $0 \in E(T)$ be the given section. Then the following assertions hold:*

(a) There exists a unique structure of commutative group-scheme on E such that the section 0 is the origin.

(b) For $N \geq 1$ an integer, the multiplication-by-N morphism $[N] : E \longrightarrow E$ is finite, flat, and locally free of rank N^2. Moreover, it sends the section 0 to itself.

Proof. For the proof we refer the reader to [KM85, Theorem 2.1.2, p. 63] and [KM85, Theorem 2.3.1, p.73]. □

Let R be a ring. We define (Ell/R) to be the category whose objects are elliptic curves E/T over R-schemes T and a morphism between two objects E_1/T_1 and E_2/T_2 is a commutative diagram

$$\begin{array}{ccc} E_2 & \xrightarrow{\alpha} & E_1 \\ {\scriptstyle\pi_2}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ T_2 & \xrightarrow{\beta} & T_1 \end{array},$$

where β is a morphism of R-schemes and $(\alpha, \pi_2) : E_2 \longrightarrow E_1 \times_{T_1} T_2$ induces an isomorphism of T_2-schemes. In case $R = \mathbb{Z}$, we will just write (Ell) instead of $(\mathrm{Ell}/\mathbb{Z})$.

Definition 3.3.3. A *moduli problem of elliptic curves*, or simply a *moduli problem*, is a contravariant functor $\mathcal{P} : (\mathrm{Ell}/R) \longrightarrow (\mathrm{Sets})$. The elements of the set $\mathcal{P}(E/T)$ are called *$\mathcal{P}$-structures of E/T*. We say that $\mathcal{P}$ is *representable* over (Ell/R) if it is representable as a functor.

Remark 3.3.4. $\mathcal{P}$ representable means that there exist an R-scheme $\mathfrak{M}(\mathcal{P})$, an elliptic curve $\mathbb{E}/\mathfrak{M}(\mathcal{P})$, and a functorial isomorphism

$$\mathcal{P}(E/S) \simeq \mathrm{Hom}_{(\mathrm{Ell}/R)}(E/S, \mathbb{E}/\mathfrak{M}(\mathcal{P})).$$

In this case, the R-scheme $\mathfrak{M}(\mathcal{P})$ is called the *fine moduli scheme of* $\mathcal{P}$.

Definition 3.3.5. Let (Sch/T) be the category of schemes over T. A moduli problem $\mathcal{P}$ is called *relatively representable over* (Ell/R) if for every elliptic curve E/T, the induced functor on (Sch/T) given by

$$W \longmapsto \mathcal{P}(E \times_T W/W)$$

is representable by a T-scheme $\mathcal{P}_{E/T}$.

Definition 3.3.6. Let P be a property of morphisms of schemes. A moduli problem $\mathcal{P}$ on (Ell/R) is *of type* P if it is relatively representable over (Ell/R) and if for every E/T, the morphism of schemes $\mathcal{P}_{E/T} \longrightarrow T$ has property P.

We proceed to describe a moduli problem associated with the principal congruence subgroup $\Gamma(N)$.

Definition 3.3.7. Let E/T be an elliptic curve. An *effective Cartier divisor* of E/T is a closed subscheme $\mathcal{D} \subset E$ which is flat over T and whose associated sheaf of ideals $\mathcal{I}_{\mathcal{D}}$ is an invertible $\mathcal{O}_E$-module.

Lemma 3.3.8. *Let E/T be an elliptic curve. Then the following assertions hold:*

(a) Every section $t \in E(T)$ defines an effective Cartier divisor proper over T.

(b) A closed subscheme $\mathcal{D} \subset E$ is an effective Cartier divisor proper over T if and only if it is finite, flat, and of finite presentation over T.

Proof. For the proof we refer the reader to [KM85, Lemma 1.2.2, 1.2.3 and 1.2.7, pp. 8–10]. □

Remark 3.3.9. Note that since any elliptic curve E/T is proper, all effective Cartier divisors are proper over T.

Let E/T be an elliptic curve, N a positive integer, and $[N] : E \longrightarrow E$ the morphism of Lemma 3.3.2 (b). We define the *subgroup scheme of the N-torsion points of E* as $E[N] \coloneqq E \times_E T$. It is obtained by base change using the morphism $[N]$ and the given section of E/T.

Definition 3.3.10. A $\Gamma(N)$-*structure* on a given elliptic curve E/T is a homomorphism of groups $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \longrightarrow E[N](T)$ such that the following identity of effective Cartier divisors holds

$$\sum_{(a,b)\in(\mathbb{Z}/N\mathbb{Z})^2} [\phi(a,b)] = E[N].$$

Here, $[\phi(a,b)]$ denotes the effective Cartier divisor induced by the section $\phi(a,b) \in E[N](T)$. The points $P = \phi(1,0)$ and $Q = \phi(0,1)$ form a *Drinfeld basis* of $E[N]$.

Let $N \geq 1$ be an integer. Consider the moduli problem $[\Gamma(N)] : (\mathrm{Ell}) \longrightarrow (\mathrm{Sets})$ given by

$$E/T \longmapsto [\Gamma(N)](E/T) := \text{ set of } \Gamma(N)\text{-structures on E/T.} \tag{3.3}$$

Proposition 3.3.11. *Let $N \geq 1$ be an integer and $[\Gamma(N)]$ the moduli problem given by* (3.3). *Then $[\Gamma(N)]$ is relatively representable, finite, and flat over* (Ell). *In particular, $[\Gamma(N)]$ is affine over* (Ell).

Proof. For the proof we refer the reader to [KM85, Theorem 5.1.1, p. 129]. □

In order to define moduli problems for the congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$, we need to consider the action of a finite group on a given moduli problem.

Definition 3.3.12. Let $\mathcal{P}$ be a moduli problem on (Ell/R) and G a finite group. We say that G *acts on* $\mathcal{P}$ if for every R-scheme T and every elliptic curve E/T, the group G acts on the set $\mathcal{P}(E/T)$ in such a way that for any morphism (α, β)

$$\begin{array}{ccc} E_2 & \xrightarrow{\alpha} & E_1 \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ T_2 & \xrightarrow{\beta} & T_1 \end{array}$$

in the category (Ell/R), the induced diagram

$$\begin{array}{ccc} G \times \mathcal{P}(E_1/T_1) & \longrightarrow & \mathcal{P}(E_1/T_1) \\ {\scriptstyle \mathrm{id}\times\mathcal{P}((\alpha,\beta))}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}((\alpha,\beta))} \\ G \times \mathcal{P}(E_2/T_2) & \longrightarrow & \mathcal{P}(E_2/T_2) \end{array}$$

commutes.

Let $\mathcal{P}$ and $\mathcal{P}'$ be two moduli problems on (Ell/R). The *simultaneous moduli problem* $(\mathcal{P}, \mathcal{P}')$ is given by $E/T \longmapsto \mathcal{P}(E/T) \times \mathcal{P}'(E/T)$. If $\mathcal{P}$ is representable by $\mathbb{E}/\mathfrak{M}(\mathcal{P})$ and $\mathcal{P}'$ is relatively representable, then $(\mathcal{P}, \mathcal{P}')$ is representable by $\mathfrak{M}(\mathcal{P}, \mathcal{P}') := \mathcal{P}'_{\mathbb{E}/\mathfrak{M}(\mathcal{P})}$.

Definition 3.3.13. Let $\mathcal{P}, \mathcal{P}'$ be two moduli problems on (Ell/R) such that a finite group G act on them. We say that $\mathcal{P}'$ *is the quotient of* $\mathcal{P}$ *by* G and write $\mathcal{P}' = \mathcal{P}/G$ if the following conditions hold:

(i) G operates trivially on $\mathcal{P}'$;

(ii) for every representable moduli problem $\mathcal{Q}$ on (Ell/R) étale over (Ell/R), the quotient scheme $\mathfrak{M}(\mathcal{Q},\mathcal{P})/G$ exists and it maps isomorphically to $\mathfrak{M}(\mathcal{Q},\mathcal{P}')$.

Proposition 3.3.14. *Let $\mathcal{P}$ be a relatively representable moduli problem affine over* (Ell/R) *and G a finite group acting on $\mathcal{P}$. Then $\mathcal{P}/G$ exists as an affine relatively representable moduli problem over* (Ell/R). *Furthermore, if $\mathcal{P}$ is finite over* (Ell/R) *and R is noetherian, then $\mathcal{P}/G$ is finite over* (Ell/R).

Proof. For the proof we refer the reader to [KM85, Theorem 7.1.3, p. 187]. □

Let (P,Q) be a Drinfeld basis of $E[N]$ given by some $\Gamma(N)$-structure. The group $G \coloneqq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on the moduli problem $[\Gamma(N)]$ by right multiplication

$$(P,Q) \longmapsto (P,Q)\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let us consider the subgroup G_0 resp. G_1 of G given by matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, respectively. Then by Proposition 3.3.14, we define the moduli problems $[\Gamma_0(N)]$ and $[\Gamma_1(N)]$ on (Ell) as follows

$$[\Gamma_0(N)] \coloneqq [\Gamma(N)]/G_0,$$
$$[\Gamma_1(N)] \coloneqq [\Gamma(N)]/G_1.$$

Remark 3.3.15. The moduli problem $[\Gamma_1(N)]$ is denoted by [bal. $\Gamma_1(N)$] in [KM85].

Finally, let us consider the following canonical construction. Let $N \geq 1$ be an integer and denote by $\Phi_N(X)$ the N-th cyclotomic polynomial. Set $\zeta_N \coloneqq X \bmod \Phi_N$ and $\mathbb{Z}[\zeta_N] \coloneqq \mathbb{Z}[X]/\langle \Phi_N(X)\rangle$. The contravariant functor

$$\begin{aligned} \mu_N^\times : (\mathrm{Sch}/\mathbb{Z}) &\longrightarrow (\mathrm{Sets}) \\ T &\longmapsto \{t \in \mathcal{O}_T(T) \mid \Phi_N(t) = 0\}, \end{aligned}$$

is representable by the affine scheme $\mathrm{Spec}(\mathbb{Z}[\zeta_N])$. Then we define the moduli problem $[\Gamma(N)]^{\mathrm{can}}$ on $(\mathrm{Ell}/\mathbb{Z}[\zeta_N])$ as follows

$$\begin{aligned}[\Gamma(N)]^{\mathrm{can}}(E/S) = \{&\Gamma(N)\text{-structures } \phi \text{ on } E/S \text{ such} \\ &\text{that } e_N(P,Q) = \zeta_N \in \mu_N^\times(S)\},\end{aligned}$$

where $e_N(\cdot,\cdot)$ is the Weil pairing (see [KM85, (2.8.5), p. 90]). It turns out that $[\Gamma(N)]^{\mathrm{can}}$ is a relatively representable moduli problem which is affine over $(\mathrm{Ell}/\mathbb{Z}[\zeta_N])$ (see [KM85, Proposition 9.1.7, p. 274]). Furthermore, it commutes with the action of the group G (see [KM85, Corollary 9.1.10, p. 276]).

The advantage of working with $[\Gamma(N)]^{\mathrm{can}}$ over $[\Gamma(N)]$ is that the fibers of $[\Gamma(N)]^{\mathrm{can}}$ over $\mathbb{Z}[\zeta_N]$ are geometrically connected. Consequently, if $\mathfrak{M}([\Gamma(N)]^{\mathrm{can}})$ is the fine moduli scheme of $[\Gamma(N)]^{\mathrm{can}}$, then after a compactification process (see [KM85, Chapter 8]) we obtain an arithmetic surface $\overline{\mathfrak{M}}([\Gamma(N)]^{\mathrm{can}})$ which is the minimal regular model of the curve $X_{\Gamma(N)}/\mathbb{Q}(\zeta_N)$. More precisely, we have the following theorem.

Theorem 3.3.16. *Let $N \geq 3$ be a composite odd square-free positive integer. Then the $\mathbb{Z}[\zeta_N]$-scheme $\overline{\mathfrak{M}}([\Gamma(N)]^{\mathrm{can}})$ is an arithmetic surface whose generic fiber is isomorphic to $X_{\Gamma(N)}/\mathbb{Q}(\zeta_N)$ with ζ_N an N-root of unity. The fiber at $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ with $\mathfrak{p} \nmid N$ is a smooth curve, whereas for the primes $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ with $\mathfrak{p} \mid N$, it consists of $p+1$ copies of smooth proper $k(\mathfrak{p})$-curves (Igusa curves) intersecting transversally at their supersingular points.*

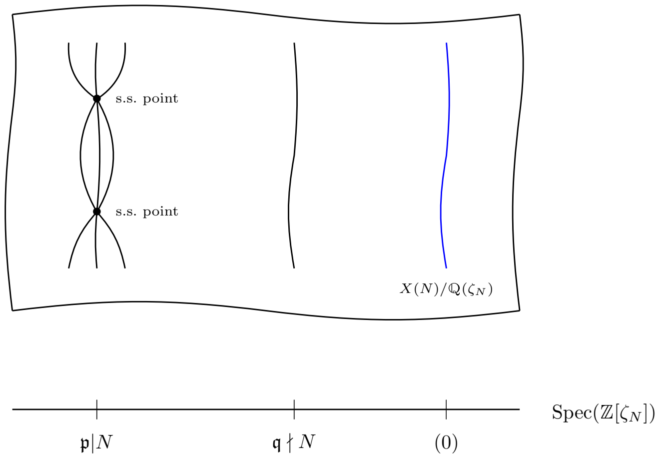

Figure 3.1: The arithmetic surface $\overline{\mathfrak{M}}([\Gamma(N)]^{\mathrm{can}})$.

Proof. For the proof, note that $\overline{\mathfrak{M}}([\Gamma(N)]^{\mathrm{can}})$ is a 2-dimensional regular $\mathbb{Z}[\zeta_N]$-scheme (see [KM85, Theorem 5.5.1, p. 144]) which is flat because the moduli problem $[\Gamma(N)]^{\mathrm{can}}$ is flat. For the properness of the structural morphism and the smoothness of the fibers at $\mathfrak{p} \nmid N$, we refer to [KM85, Summarizing table, p. 305]. Then, by [Liu02, Theorem 3.16, p. 353], we have that $\overline{\mathfrak{M}}([\Gamma(N)]^{\mathrm{can}})$ is an arithmetic surface over $\mathrm{Spec}(\mathbb{Z}[\zeta_N])$. For the description of the fibers at $\mathfrak{p}|N$, see [KM85, Theorem 13.7.6, p. 427]. $\square$

Remark 3.3.17. In fact, the arithmetic surface $\overline{\mathfrak{M}}([\Gamma(N)]^{\mathrm{can}})$ is minimal. Indeed, note that on the fibers at primes $\mathfrak{p} \nmid N$, if there are projective lines, they would have self-intersection equal to zero because the fiber is smooth. On the other hand, on the fibers at primes $\mathfrak{p}|N$, the number of Igusa curves living in this fibers is $p+1$, so the self-intersection is less than $-p$; hence, not -1.

Theorem 3.3.18. *Let $[\Gamma_1(N)]^{\mathrm{can}} = [\Gamma(N)]^{\mathrm{can}}/G_1$ with $N = N'qr$ an odd square-free integer with $q, r > 4$ two different prime numbers. Then the $\mathbb{Z}[\zeta_N]$-scheme $\overline{\mathfrak{M}}([\Gamma_1(N)]^{\mathrm{can}})$ is an arithmetic surface whose generic fiber is isomorphic to $X_{\Gamma_1(N)}/\mathbb{Q}(\zeta_N)$ with ζ_N an N-th root of unity. The fiber at $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ with $\mathfrak{p} \nmid N$ is a smooth curve, whereas for a prime $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ with $\mathfrak{p} \mid N$, it consists of 2 copies of smooth proper $k(\mathfrak{p})$-curves which are geometrically connected that intersect transversally at their supersingular points. Furthermore, $\overline{\mathfrak{M}}([\Gamma_1(N)]^{\mathrm{can}})$ is minimal.*

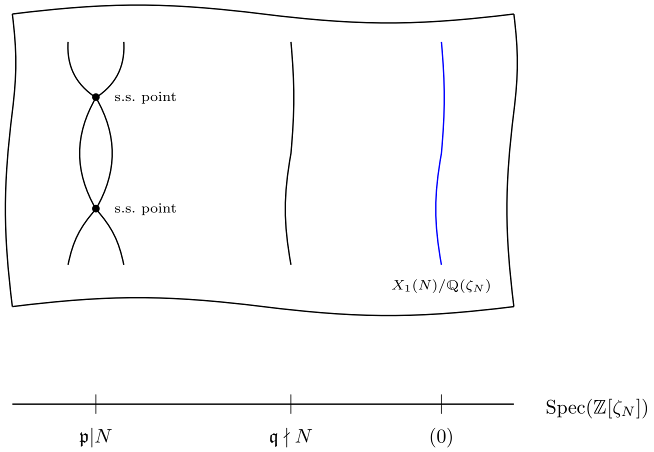

Figure 3.2: The arithmetic surface $\overline{\mathfrak{M}}([\Gamma_1(N)]^{\mathrm{can}})$.

Proof. The proof is similar to the proof of Theorem 3.3.16. For the description of the fibers at $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ with $\mathfrak{p}|N$, we refer the reader to [KM85, Theorem 13.11.4, p. 449]. □

Remark 3.3.19. The minimal regular model associated to the congruence subgroup $\Gamma_0(N)$ is given by the minimal resolution of the coarse moduli scheme of the moduli problem $[\Gamma_0(N)]$ provided that N is square-free and relatively prime to 6 such that $g_{\Gamma_0(N)} \geq 1$. Thus one obtains an arithmetic surface $\overline{\mathfrak{M}}([\Gamma_0(N)])$ over $\mathbb{Z}$. For a detailed description of this $\mathbb{Z}$-scheme, we refer the reader to [AU97, p. 62].

Notation 3.3.20. In the sequel, we let $\mathcal{X}_\Gamma$ denote one of the $\mathcal{O}_K$-schemes

$$\overline{\mathfrak{M}}([\Gamma_0(N)]), \quad \overline{\mathfrak{M}}([\Gamma_1(N)]^{\mathrm{can}}), \quad \text{or} \quad \overline{\mathfrak{M}}([\Gamma(N)]^{\mathrm{can}})$$

with K chosen as appropriate and N fulfilling the required conditions for existence in each case. The scheme $\mathcal{X}_\Gamma/\mathcal{O}_K$ is the *minimal regular model of the curve* X_Γ/K associated to Γ.

3.4 Arithmetic self-intersection of the relative dualizing sheaf

In what follows, we let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$ and denote by $\mathcal{X}_\Gamma/\mathcal{O}_K$ the minimal regular model associated to Γ. We will write X_Γ/K for the generic fiber $(\mathcal{X}_\Gamma)_\eta$.

Consider the extension of scalars $\widehat{\mathrm{Pic}}(\mathcal{X}_\Gamma) \otimes \mathbb{Q}$. It can be verified that the intersection pairing $(\cdot,\cdot)_{\mathrm{Ar}}$ extends to $\left(\widehat{\mathrm{Pic}}(\mathcal{X}_\Gamma) \otimes \mathbb{Q}\right) \times \left(\widehat{\mathrm{Pic}}(\mathcal{X}_\Gamma) \otimes \mathbb{Q}\right)$. In the next lemma we regard $\overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ as an element in $\widehat{\mathrm{Pic}}(\mathcal{X}_\Gamma) \otimes \mathbb{Q}$.

Lemma 3.4.1. *Let H_q be the horizontal divisor defined by the cusp $q \in C_\Gamma$ regarded as a K-rational point of X_Γ/K. Suppose that $q = 0$ or ∞. Then there exists a vertical divisor $V_q \in \mathrm{Z}^1(\mathcal{X}_\Gamma) \otimes \mathbb{Q}$ such that the identity*

$$\left(\overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K} \otimes \overline{\mathcal{O}_{\mathcal{X}}(H_q)}^{\otimes -(2g_\Gamma - 2)} \otimes \overline{\mathcal{O}_{\mathcal{X}}(V_q)}\,,\, \overline{\mathcal{O}_{\mathcal{X}}(V)}\right)_{\mathrm{Ar}} = 0 \tag{3.4}$$

holds for all vertical divisors V of $\mathcal{X}_\Gamma/\mathcal{O}_K$.

Proof. For the proof we will consider only the case $\Gamma = \Gamma(N)$. For the other two cases we refer the reader to [AU97, §4, p. 60] and [May14, Proposition 7.5, pp. 38–39].

Let $p|N$ be a prime number and $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ a prime ideal over p. We set $r_\mathfrak{p} := p + 1$ and let $s_\mathfrak{p}$ be the number of supersingular points at $\mathfrak{p}$, namely, we have

$$s_\mathfrak{p} := \frac{p-1}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N/p)].$$

Let $C_{1,\mathfrak{p}}, \ldots, C_{r_\mathfrak{p},\mathfrak{p}}$ be the irreducible components of the fiber $(\mathcal{X}_\Gamma)_\mathfrak{p}$ and denote by $C_{0,\mathfrak{p}}$ resp. $C_{\infty,\mathfrak{p}}$ the irreducible component intersected by the horizontal divisor H_0 and H_∞, respectively (see [Liu02, Corollary 1.32, p. 388]). By virtue of [KM85, Theorem 13.9.3, p. 435], we have $C_{0,\mathfrak{p}} \neq C_{\infty,\mathfrak{p}}$.
Define

$$V_0 := -\sum_{\mathfrak{p}|N} \frac{2(g_\Gamma - 1)}{r_\mathfrak{p} s_\mathfrak{p}} C_{0,\mathfrak{p}} \quad \text{and} \quad V_\infty := -\sum_{\mathfrak{p}|N} \frac{2(g_\Gamma - 1)}{r_\mathfrak{p} s_\mathfrak{p}} C_{\infty,\mathfrak{p}} \tag{3.5}$$

(see Figure 3.3 on the next page). We claim that with these divisors, the identity (3.4) is satisfied. Let $\widehat{\mathcal{K}}$ be an Arakelov divisor that corresponds to $\overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K}$. First of all, note that

$$(\widehat{\mathcal{K}}, C_{i,\mathfrak{p}})_{\mathrm{fin}} = \frac{2g_\Gamma - 2}{r_\mathfrak{p}} \log(\#k(\mathfrak{p})).$$

Indeed, this is a consequence of [Liu02, Proposition 1.35, p. 389] and the fact that the components $C_{j,\mathfrak{p}}$ $(j = 1, \ldots, r_{\mathfrak{p}})$ are all isomorphic.

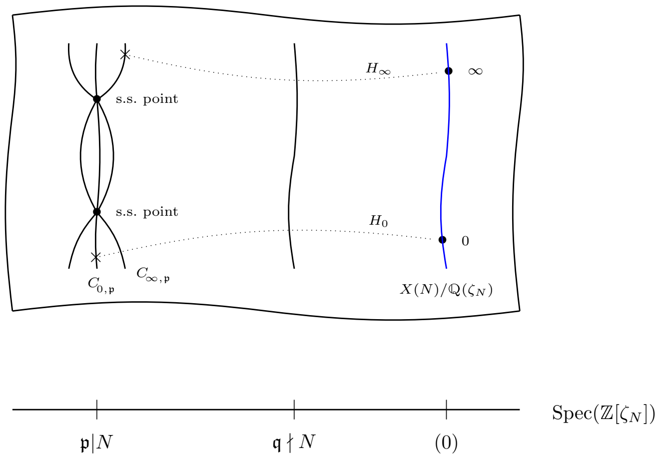

Figure 3.3: Construction of the distinguished divisors V_0 and V_∞.

Secondly, using [Liu02, Proposition 1.21, p. 384] one can easily deduce the identities

$$(C_{q_1,\mathfrak{p}}, C_{q_2,\mathfrak{p}})_{\mathrm{fin}} = \begin{cases} s_{\mathfrak{p}} \cdot \log(\#k(\mathfrak{p})), & q_1 \neq q_2; \\ -(r_{\mathfrak{p}} - 1)s_{\mathfrak{p}} \cdot \log(\#k(\mathfrak{p})), & q_1 = q_2; \end{cases} \tag{3.6}$$

where $q_1, q_2 \in \{0, \infty\}$. Thirdly, a direct calculation gives the following:

(i) If $V = V_{\mathfrak{p}}$ with $\mathfrak{p} \nmid N$, then we have

$$\begin{aligned} (\widehat{\mathcal{K}}, V)_{\mathrm{fin}} &= (2g_\Gamma - 2)\log(\#k(\mathfrak{p})), \\ (H_q, V)_{\mathrm{fin}} &= \log(\#k(\mathfrak{p})), \\ (V_q, V)_{\mathrm{fin}} &= 0. \end{aligned}$$

(ii) If $V = C_{q,\mathfrak{p}}$ with $q \in \{0, \infty\}$ and $\mathfrak{p}|N$, then we have

$$\begin{aligned} (\widehat{\mathcal{K}}, V)_{\mathrm{fin}} &= \frac{2g_\Gamma - 2}{r_{\mathfrak{p}}}\log(\#k(\mathfrak{p})), \\ (H_q, V)_{\mathrm{fin}} &= \log(\#k(\mathfrak{p})), \\ (V_q, V)_{\mathrm{fin}} &= \sum_{\mathfrak{q}|N} \frac{2(g_\Gamma - 1)}{r_{\mathfrak{q}} s_{\mathfrak{q}}}(C_{q,\mathfrak{q}}, C_{q,\mathfrak{p}})_{\mathrm{fin}} \\ &= \frac{2(g_\Gamma - 1)(r_{\mathfrak{p}} - 1)}{r_{\mathfrak{p}}}\log(\#k(\mathfrak{p})). \end{aligned}$$

(iii) If $V = C_{q_2,\mathfrak{p}}$ with $q_1, q_2 \in \{0, \infty\}$, $q_1 \neq q_2$, and $\mathfrak{p}|N$; then

$$
\begin{aligned}
(\widehat{\mathcal{K}}, V)_{\mathrm{fin}} &= \frac{2g_{\Gamma} - 2}{r_{\mathfrak{p}}} \log(\#k(\mathfrak{p})), \\
(H_{q_1}, V)_{\mathrm{fin}} &= 0, \\
(V_{q_1}, V)_{\mathrm{fin}} &= -\sum_{\mathfrak{q}|N} \frac{2(g_{\Gamma} - 1)}{r_{\mathfrak{q}} s_{\mathfrak{q}}} (C_{q_1,\mathfrak{q}}, C_{q_2,\mathfrak{p}})_{\mathrm{fin}} \\
&= -\frac{2(g_{\Gamma} - 1)}{r_{\mathfrak{p}}} \log(\#k(\mathfrak{p})).
\end{aligned}
$$

(iv) If $V = C_{q_2,\mathfrak{p}}$ with $q_1 \in \{0, \infty\}$, $q_1 \neq q_2$, and $\mathfrak{p}|N$; then

$$
\begin{aligned}
(\widehat{\mathcal{K}}, V)_{\mathrm{fin}} &= \frac{2g_{\Gamma} - 2}{r_{\mathfrak{p}}} \log(\#k(\mathfrak{p})), \\
(H_{q_1}, V)_{\mathrm{fin}} &= 0, \\
(V_{q_1}, V)_{\mathrm{fin}} &= -\sum_{\mathfrak{q}|N} \frac{2(g_{\Gamma} - 1)}{r_{\mathfrak{q}} s_{\mathfrak{q}}} (C_{q_1,\mathfrak{q}}, C_{q_2,\mathfrak{p}})_{\mathrm{fin}} \\
&= -\frac{2(g_{\Gamma} - 1)}{r_{\mathfrak{p}}} \log(\#k(\mathfrak{p})).
\end{aligned}
$$

Since

$$
\begin{aligned}
\Big(\overline{\omega}_{\mathcal{X}_{\Gamma}/\mathcal{O}_K} \otimes \overline{\mathcal{O}_{\mathcal{X}}(H_q)}^{\otimes -(2g_{\Gamma}-2)} \otimes \overline{\mathcal{O}_{\mathcal{X}}(V_q)}\,,\, \overline{\mathcal{O}_{\mathcal{X}}(V)}\Big)_{\mathrm{Ar}} &= \Big(\overline{\omega}_{\mathcal{X}_{\Gamma}/\mathcal{O}_K}\,,\, \overline{\mathcal{O}_{\mathcal{X}}(V)}\Big)_{\mathrm{fin}} \\
- (2g_{\Gamma} - 2)\Big(\overline{\mathcal{O}_{\mathcal{X}}(H_q)}\,,\, \overline{\mathcal{O}_{\mathcal{X}}(V)}\Big)_{\mathrm{fin}} &+ \Big(\overline{\mathcal{O}_{\mathcal{X}}(V_q)}\,,\, \overline{\mathcal{O}_{\mathcal{X}}(V)}\Big)_{\mathrm{fin}},
\end{aligned}
$$

it can be easily verified that (3.4) holds for the cases (i), (ii), (iii), and (iv). Since an arbitrary vertical divisor is a linear combination of $V_{\mathfrak{p}}$ and $C_{q,\mathfrak{p}}$, the result follows by the bilinearity of the pairing $(\cdot,\cdot)_{\mathrm{Ar}}$. $\square$

Proposition 3.4.2. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, and let $\mathcal{X}_{\Gamma}/\mathcal{O}_K$ be the minimal regular model associated to Γ. Suppose that $g_{\Gamma} \geq 2$. Then we have*

$$
\begin{aligned}
\overline{\omega}^2_{\mathcal{X}_{\Gamma}/\mathcal{O}_K} = \frac{2g_{\Gamma}(V_0, V_{\infty})_{\mathrm{fin}} - (V_0, V_0)_{\mathrm{fin}} - (V_{\infty}, V_{\infty})_{\mathrm{fin}}}{2(g_{\Gamma} - 1)}& \\
- 2g_{\Gamma}(g_{\Gamma} - 1) \sum_{\sigma : K \hookrightarrow \mathbb{C}} g^{\Gamma}_{\mathrm{can}}(0^{\sigma}, \infty^{\sigma}),&
\end{aligned}
$$

where 0^{σ} resp. ∞^{σ} denotes the image under σ of 0 and ∞, respectively.

Remark 3.4.3. In the sequel, the first resp. second terms of the previous identity will be referred as the *geometric and analytic contribution of* $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$, respectively.

Proof. For the proof, we set

$$\overline{\mathcal{L}}_q := \overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K} \otimes \overline{\mathcal{O}_{\mathcal{X}}(H_q)}^{\otimes -(2g_\Gamma - 2)} \otimes \overline{\mathcal{O}_{\mathcal{X}}(V_q)},$$

where $q \in C_\Gamma$ with $q = 0$ or ∞, and H_q resp. V_q is the horizontal and vertical divisor of Lemma 3.4.1, respectively. For simplicity let us write $D^2 = (D, D)_{\mathrm{Ar}}$.

Let us consider the line bundle $L_q := \mathcal{L}_q \otimes_{\mathcal{O}_K} K$ on X_Γ/K, i.e., the pullback of $\mathcal{L}_q$ to the generic fiber. By the Manin–Drinfeld theorem (see [Dri73]), we know that L_q is a torsion point in the Jacobian $\mathrm{Jac}(X_\Gamma)(K)$; therefore, the Faltings–Hriljac theorem (see [Hri85]) implies that

$$\overline{\mathcal{L}}_q^2 = -2[K : \mathbb{Q}] h_{\mathrm{NT}}(L_q),$$

where $h_{\mathrm{NT}}(\cdot)$ denotes the Néron–Tate height (see [BG06, Section 9.4, p. 294]). Since $h_{\mathrm{NT}}(L_q) = 0$, we obtain $\overline{\mathcal{L}}_q^2 = 0$, i.e., we have

$$\Big(\overline{\mathcal{L}}_q \,, \overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K} \otimes \overline{\mathcal{O}_{\mathcal{X}}(H_q)}^{\otimes -(2g_\Gamma - 2)}\Big)_{\mathrm{Ar}} + \Big(\overline{\mathcal{L}}_q, \overline{\mathcal{O}_{\mathcal{X}}(V_q)}\Big)_{\mathrm{Ar}} = 0. \tag{3.7}$$

Note that the last term vanishes by (3.4). Now, expanding the remaining term on the left hand side of (3.7), we obtain

$$\begin{aligned}
\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K} &= (2g_\Gamma - 2)\Big(\overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K} \,, \overline{\mathcal{O}_{\mathcal{X}}(H_q)}\Big)_{\mathrm{Ar}} + (2g_\Gamma - 2)\Big(\overline{\mathcal{O}_{\mathcal{X}}(H_q)} \,, \overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K}\Big)_{\mathrm{Ar}} \\
&\quad - (2g_\Gamma - 2)^2 \overline{\mathcal{O}_{\mathcal{X}}(H_q)}^2 - \Big(\overline{\mathcal{O}_{\mathcal{X}}(V_q)} \,, \overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K}\Big)_{\mathrm{Ar}} \\
&\quad + (2g_\Gamma - 2)\Big(\overline{\mathcal{O}_{\mathcal{X}}(V_q)} \,, \overline{\mathcal{O}_{\mathcal{X}}(H_q)}\Big)_{\mathrm{Ar}}.
\end{aligned}$$

Using (3.4) with $V = V_q$, we can deduce

$$-\overline{\mathcal{O}_{\mathcal{X}}(V_q)}^2 = \Big(\overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K} \otimes \overline{\mathcal{O}_{\mathcal{X}}(H_q)}^{\otimes -(2g_\Gamma - 2)} \,, \overline{\mathcal{O}_{\mathcal{X}}(V_q)}\Big)_{\mathrm{Ar}};$$

hence, we have

$$\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K} = 2(2g_\Gamma - 2)\Big(\overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K} \,, \overline{\mathcal{O}_{\mathcal{X}}(H_q)}\Big)_{\mathrm{Ar}} - (2g_\Gamma - 2)^2 \overline{\mathcal{O}_{\mathcal{X}}(H_q)}^2 + \overline{\mathcal{O}_{\mathcal{X}}(V_q)}^2.$$

This amounts to

$$\begin{aligned}
\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K} &= 2(2g_\Gamma - 2)\Big(\overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K} \otimes \overline{\mathcal{O}_{\mathcal{X}}(H_q)} \,, \overline{\mathcal{O}_{\mathcal{X}}(H_q)}\Big)_{\mathrm{Ar}} \\
&\qquad - 2g_\Gamma(2g_\Gamma - 2)\overline{\mathcal{O}_{\mathcal{X}}(H_q)}^2 + \overline{\mathcal{O}_{\mathcal{X}}(V_q)}^2,
\end{aligned}$$

and since $\Big(\overline{\omega}_{\mathcal{X}_\Gamma/\mathcal{O}_K} \otimes \overline{\mathcal{O}_{\mathcal{X}}(H_q)} \,, \overline{\mathcal{O}_{\mathcal{X}}(H_q)}\Big)_{\mathrm{Ar}} = 0$ by [Lan88, Corollary 5.6, p. 101], we have

$$\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K} = -4g_\Gamma(g_\Gamma - 1)\overline{\mathcal{O}_{\mathcal{X}}(H_0)}^2 + \overline{\mathcal{O}_{\mathcal{X}}(V_0)}^2,$$

$$\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K} = -4g_\Gamma(g_\Gamma - 1)\overline{\mathcal{O}_{\mathcal{X}}(H_\infty)}^2 + \overline{\mathcal{O}_{\mathcal{X}}(V_\infty)}^2.$$

Adding these identities, we get

$$\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K} = -2g_\Gamma(g_\Gamma - 1)\Big(\overline{\mathcal{O}_{\mathcal{X}}(H_0)}^2 + \overline{\mathcal{O}_{\mathcal{X}}(H_\infty)}^2\Big) + \frac{1}{2}\Big(\overline{\mathcal{O}_{\mathcal{X}}(V_0)}^2 + \overline{\mathcal{O}_{\mathcal{X}}(V_\infty)}^2\Big). \tag{3.8}$$

Now, let us consider

$$\mathcal{M} := H_\infty - H_0 + \frac{V_0 - V_\infty}{2g_\Gamma - 2}.$$

The divisor $\mathcal{M}$ satisfies $(\mathcal{M}, V)_{\mathrm{Ar}} = 0$, for all vertical divisors V, and the restriction of $\mathcal{M}$ to X_Γ/K has support in the cusps. Therefore, $\mathcal{M}^2 = 0$. If we set

$$\mathcal{H} = H_\infty - H_0 \quad \text{and} \quad \mathcal{V} = \frac{V_0 - V_\infty}{2g_\Gamma - 2},$$

then $\mathcal{H} = \mathcal{M} - \mathcal{V}$ and with this we note that

$$\mathcal{H}^2 = \mathcal{M}^2 - 2(\mathcal{M}, \mathcal{V})_{\mathrm{Ar}} + \mathcal{V}^2 = \mathcal{V}^2.$$

Therefore, we have

$$H_\infty^2 + H_0^2 = 2(H_\infty, H_0)_{\mathrm{Ar}} + \frac{V_0^2 - 2(V_0, V_\infty)_{\mathrm{Ar}} + V_\infty^2}{(2g_\Gamma - 2)^2}; \tag{3.9}$$

and hence, by (3.8), we obtain

$$\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K} = -4g_\Gamma(g_\Gamma - 1)(H_\infty, H_0)_{\mathrm{inf}} + \frac{2g_\Gamma(V_0, V_\infty)_{\mathrm{fin}} - (V_0, V_0)_{\mathrm{fin}} - (V_\infty, V_\infty)_{\mathrm{fin}}}{2(g_\Gamma - 1)},$$

because $(H_\infty, H_0)_{\mathrm{fin}} = 0$. This concludes the proof. $\square$

Chapter 4

Analytic contribution of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$

In this chapter we determine the analytic contribution of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ given in Proposition 3.4.2. To do so, we extend the methods and results of [AU97] to the subgroup $\Gamma(N)$.

In Section 4.1, we determine the canonical Green's function $g^{\Gamma}_{\mathrm{can}}(0^\sigma,\infty^\sigma)$ in terms of the level N, for the congruence subgroups $\Gamma=\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$. For this, we first show that the problem reduces to determine the constant $\mathscr{R}^{\Gamma}_{\infty}$ in terms of N, and then we explain how this constant is the sum of essentially three contributions: an elliptic one, denoted by $\mathscr{R}^{\mathrm{ell}}_{\infty}$, an hyperbolic one, denoted by $\mathscr{R}^{\mathrm{hyp}}_{\infty}$, and a parabolic one, denoted by $\mathscr{R}^{\mathrm{par}}_{\infty}$. The statements of the main results of this chapter are given in this section.

In Section 4.2, we introduce the zeta function associated to a matrix of a congruence subgroup of fixed trace. For the case of the principal congruence subgroup $\Gamma(N)$, we prove that this zeta function is essentially the zeta function of an ideal class in the narrow-ray class group of a real quadratic field. The material here is a preparation for the next section.

In Section 4.3, we determine the hyperbolic contribution $\mathscr{R}^{\mathrm{hyp}}_{\infty}$ for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$.

In Section 4.4, we determine the parabolic contribution $\mathscr{R}^{\mathrm{par}}_{\infty}$ for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$.

The main references for this part are [AU97], [Hej83], [May14], and [Zag81a].

4.1 Overview and statement of main results

Let $N\geq 3$ be an odd square-free integer and $\Gamma\subset\mathrm{SL}_2(\mathbb{Z})$ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$. The goal of this section is to determine the value of

$$-\frac{2g_{\Gamma}(g_{\Gamma}-1)}{\varphi(N)}\sum_{\sigma:K\hookrightarrow\mathbb{C}} g_{\mathrm{can}}^{\Gamma}(0^{\sigma},\infty^{\sigma}) \tag{4.1}$$

(c.f. Proposition 3.4.2) in terms of the level N. For this we claim that for each embedding $\sigma : K \longrightarrow \mathbb{C}$, we have $0^{\sigma} = \infty$ and $\infty^{\sigma} = 0_v$, if $\Gamma = \Gamma_1(N)$, whereas $0^{\sigma} = 0_v$ and $\infty^{\sigma} = \infty$, if $\Gamma = \Gamma(N)$ (for the proof, see Proposition 5.1.6); here, 0_v is given by Notation 2.2.6. Consequently, to achieve our goal, it suffices to consider only the following three cases of the canonical Green's function evaluated at different cusps

$$g_{\mathrm{can}}^{\Gamma_0(N)}(0,\infty),\quad g_{\mathrm{can}}^{\Gamma_1(N)}(0_v,\infty),\ \text{ and }\quad g_{\mathrm{can}}^{\Gamma(N)}(0_v,\infty). \tag{4.2}$$

The next theorem gives a satisfactory answer for our purposes.

Theorem 4.1.1. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_{\Gamma} > 1$. Then the following identity holds for the three cases of* (4.2)

$$g_{\mathrm{can}}^{\Gamma}(q_1,q_2) = 4\pi\mathscr{C}_{q_1q_2}^{\Gamma} + \frac{4\pi}{v_{\Gamma}} - 8\pi\mathscr{R}_{\infty}^{\Gamma} + O\left(\frac{1}{g_{\Gamma}}\right), \tag{4.3}$$

where $\mathscr{C}_{q_1q_2}^{\Gamma}$ is defined by (1.14) *and $\mathscr{R}_{\infty}^{\Gamma}$ is given by* (1.20). *Furthermore, the following assertions hold:*

(a) If $\Gamma = \Gamma_0(N)$, then we have

$$\begin{aligned}\mathscr{R}_{\infty}^{\Gamma} = &-\frac{v_{\Gamma}^{-1}}{2g_{\Gamma}}\lim_{s\to1}\left(\frac{Z_{\Gamma}'}{Z_{\Gamma}}(s)-\frac{1}{s-1}\right)+\frac{1-\log(4\pi)}{4\pi g_{\Gamma}}+\frac{\mathscr{C}_{\infty\infty}^{\Gamma}}{g_{\Gamma}}\\ &+\frac{d(N)}{v_{\Gamma}g_{\Gamma}}\left[C_1-\mathscr{C}-\frac{\gamma_{\mathrm{EM}}}{2}+\frac{1}{2}\sum_{p|N}\frac{2p+1}{p+1}\log(p)-\frac{1}{2}\frac{\varphi(N)}{Nd(N)}-\frac{1}{4}\log(N)\right]\\ &+\frac{v_{\Gamma}^{-1}}{g_{\Gamma}}\left(\prod_{p|N}\left(1+\left(\frac{-1}{p}\right)\right)\right)\left[\frac{1}{4}\sum_{p|N}\frac{p}{1+p}\log(p)+C_0\right]\\ &+\frac{v_{\Gamma}^{-1}}{g_{\Gamma}}\left(\prod_{p|N}\left(1+\left(\frac{-3}{p}\right)\right)\right)\left[\frac{1}{3}\sum_{p|N}\frac{p}{1+p}\log(p)+C_1\right].\end{aligned}$$

(b) If $\Gamma = \Gamma_1(N)$, then we have

$$\begin{aligned}\mathscr{R}_{\infty}^{\Gamma} = &-\frac{v_{\Gamma}^{-1}}{2g_{\Gamma}}\lim_{s\to1}\left(\frac{Z_{\Gamma}'}{Z_{\Gamma}}(s)-\frac{1}{s-1}\right)+\frac{1-\log(4\pi)}{4\pi g_{\Gamma}}+\frac{\mathscr{C}_{\infty\infty}^{\Gamma}}{g_{\Gamma}}\\ &+\frac{\varphi(N)}{v_{\Gamma}g_{\Gamma}}\prod_{p|N}\left(1+\frac{2}{p}\right)\left[C_1-\mathscr{C}-\frac{\gamma_{\mathrm{EM}}}{2}+\frac{1}{2}\sum_{p|N}\frac{2p+1}{p+1}\log(p)\right.\end{aligned}$$

$$-\frac{1}{2}\left(1+\sum_{j=1}^{\omega(N)} 2^{\omega(N)-j}\sum_{\substack{d|N\\ \omega(d)=j}} d\log(d)\right)\prod_{p|N}\frac{1}{p+2}\Bigg].$$

(c) If $\Gamma = \Gamma(N)$, then we have

$$\mathscr{R}^{\Gamma}_{\infty} = -\frac{v_{\Gamma}^{-1}}{2g_{\Gamma}}\lim_{s\to 1}\left(\frac{Z'_{\Gamma}}{Z_{\Gamma}}(s)-\frac{1}{s-1}\right)+\frac{1-\log(4\pi)}{4\pi g_{\Gamma}}+\frac{\mathscr{C}^{\Gamma}_{\infty\infty}}{g_{\Gamma}}$$
$$+\frac{12}{\pi g_{\Gamma} N}\left[C_1-\mathscr{C}-\frac{\gamma_{\mathrm{EM}}}{2}+\log(N)\right].$$

Here, $Z_{\Gamma}(s)$ resp. γ_{EM} denotes the Selberg zeta function and the Euler–Mascheroni constant, respectively, and C_0, C_1 are constants that do not depend on N.

The previous theorem together with Theorem 2.2.12 and the corollaries 2.3.7 and 2.3.8, allow the study of the asymptotic behaviour of (4.1) as the level N tends to infinity. As a result, we will deduce in Theorem 5.2.3 the following asymptotic expansion

$$-\frac{2g_{\Gamma(N)}(g_{\Gamma(N)}-1)}{\varphi(N)}\sum_{\sigma:K\hookrightarrow\mathbb{C}} g^{\Gamma(N)}_{\mathrm{can}}(0^{\sigma},\infty^{\sigma}) = 4g_{\Gamma(N)}\log(N)+o(g_{\Gamma(N)}\log(N)),$$

as $N\to\infty$. In what follows, we will discuss the proof of Theorem 4.1.1.

Proof of the identity (4.3)

First of all, by the spectral interpretation of $g^{\Gamma}_{\mathrm{can}}(q_1,q_2)$ (see Theorem 1.6.13), we have

$$g^{\Gamma}_{\mathrm{can}}(q_1,q_2) = 4\pi\mathscr{C}^{\Gamma}_{q_1q_2}+\frac{4\pi}{v_{\Gamma}}-4\pi\left(\mathscr{R}^{\Gamma}_{q_1}+\mathscr{R}^{\Gamma}_{q_2}\right)$$
$$+4\pi\lim_{s\to 1}\left(\int\limits_{X(\Gamma)\times X(\Gamma)} G^{\Gamma}_s(z,w)\mu_{\mathrm{can}}(z)\mu_{\mathrm{can}}(w)-\frac{v_{\Gamma}^{-1}}{s(s-1)}\right).$$

The next proposition reveals the O-term in the identity (4.3).

Proposition 4.1.2. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where N is a positive integer such that $g_{\Gamma}>1$. Then the following identity holds*

$$\lim_{s\to 1}\left(\int\limits_{X(\Gamma)\times X(\Gamma)} G^{\Gamma}_s(z,w)\mu_{\mathrm{can}}(z)\mu_{\mathrm{can}}(w)-\frac{v_{\Gamma}^{-1}}{s(s-1)}\right)=O\left(\frac{1}{g_{\Gamma}}\right),$$

where $G^{\Gamma}_s(z,w)$ is the automorphic Green's function given by Definition 1.6.12.

Proof. (Sketch.) For the proof, let $X = X(\Gamma)$. Then we have

$$\lim_{s\to 1}\left(G_s^\Gamma(z,w) - \frac{v_\Gamma^{-1}}{s(s-1)}\right) = \frac{1}{4\pi} g_{\mathrm{hyp}}(z,w),$$

where $g_{\mathrm{hyp}}(z,w)$ is the hyperbolic Green's function given by [JK06b, (3), p. 684]. Using [JK06b, Lemma 3.7, p. 690] and [JK06b, Proposition 4.7, p. 694], we obtain

$$\int_{X\times X} g_{\mathrm{hyp}}(z,w)\mu_{\mathrm{can}}(z)\mu_{\mathrm{can}}(w) \leq \frac{\pi(d_X+1)^2 v_\Gamma}{g_\Gamma^2 \lambda_{X,1}},$$

where $\lambda_{X,1}$ denotes the smallest non-zero eigenvalue of $\Delta_{\mathrm{hyp},0}$ on X and d_X is given by

$$d_X \coloneqq \sup_{z\in X}\left|\frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)}\right|$$

with $\mu_{\mathrm{shyp}}(z) = \mu_{\mathrm{hyp}}(z)/v_\Gamma$. Since $g_\Gamma > 1$, there are positive constants C_3 and c satisfying $\lambda_{X,1} \geq c$ (see [JK06b, Lemma 5.3, p. 696]) and $d_X \leq C_3$ (see [JK06b, Lemma 5.9, 698]). Using the fact that $g_\Gamma / v_\Gamma = O(1)$, the result follows. □

The next proposition reduces the task of determining the constants $\mathscr{R}_{q_1}^\Gamma$ and $\mathscr{R}_{q_2}^\Gamma$ to determine $\mathscr{R}_\infty^\Gamma$ only, provided that Γ is one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$.

Proposition 4.1.3. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_\Gamma > 1$. Then the following assertions hold:*

(a) If $\Gamma = \Gamma_0(N)$, then we have $\mathscr{R}_0^\Gamma = \mathscr{R}_\infty^\Gamma$.

(b) If $\Gamma = \Gamma_1(N)$, then we have $\mathscr{R}_{0_v}^\Gamma = \mathscr{R}_\infty^\Gamma$.

(c) If $\Gamma = \Gamma(N)$, then we have $\mathscr{R}_{0_v}^\Gamma = \mathscr{R}_\infty^\Gamma$.

To prove this proposition, we need the following lemma.

Lemma 4.1.4. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index with $g_\Gamma > 1$ and suppose that α is a 2×2-matrix with entries in $\mathbb{Q}$ and* $\det(\alpha) > 0$ *satisfying $\alpha^{-1}\Gamma\alpha = \Gamma$. Let $F_\Gamma(z)$ be the Arakelov metric on $X(\Gamma)$ defined by* (1.19). *Then the following identity holds*

$$F_\Gamma(\alpha z) = F_\Gamma(z).$$

Proof. For the proof, let us write

$$(f[\alpha]_2)(z) \coloneqq \det(\alpha)(cz+d)^{-2} f(\alpha z) \tag{4.4}$$

for any function $f : \mathbb{H} \longrightarrow \mathbb{C}$. Then, under the hypothesis of the lemma, we have that $\cdot[\alpha]_2$ acts on the space of cusp forms $\mathcal{S}_2(\Gamma)$ of weight 2 (see [DS05, Chapter 5]). Moreover, it defines an isometry on $\mathcal{S}_2(\Gamma)$ with respect to the Petersson inner product. Consequently, if $\{f_j\}_{j=1}^{g_\Gamma}$ is an orthonormal basis of $\mathcal{S}_2(\Gamma)$, then $\{f_j[\alpha]_2\}_{j=1}^{g_\Gamma}$ is as well; hence, we have

$$\sum_{j=1}^{g_\Gamma} |f_j(z)|^2 = \sum_{j=1}^{g_\Gamma} |(f_j[\alpha]_2)(z)|^2.$$

Using (4.4) on the right hand side of the previous identity, we obtain the identity

$$\sum_{j=1}^{g_\Gamma} |f_j(z)|^2 = \left(\frac{\det(\alpha)}{|cz+d|^2}\right)^2 \sum_{j=1}^{g_\Gamma} |f_j(\alpha z)|^2.$$

If we multiply both sides by $\mathrm{Im}(z)^2/g_\Gamma$, then we have

$$\begin{aligned}\frac{\mathrm{Im}(z)^2}{g_\Gamma}\sum_{j=1}^{g_\Gamma} |f_j(z)|^2 &= \frac{\mathrm{Im}(z)^2}{g_\Gamma}\left(\frac{\det(\alpha)}{|cz+d|^2}\right)^2 \sum_{j=1}^{g_\Gamma} |f_j(\alpha z)|^2\\ &= \frac{\mathrm{Im}(\alpha z)^2}{g_\Gamma}\sum_{j=1}^{g_\Gamma} |f_j(\alpha z)|^2,\end{aligned}$$

where in the last identity we used the equality $\mathrm{Im}(\alpha z) = \det(\alpha)\mathrm{Im}(z)/|cz+d|^2$. Hence, we have $F_\Gamma(\alpha z) = F_\Gamma(z)$. This concludes the proof. $\square$

Proof of Proposition 4.1.3. First of all, recall that the constant $\mathscr{R}_q^\Gamma$ is defined by

$$\mathscr{R}_q^\Gamma = \lim_{s\to 1}\left(\mathcal{R}_q^\Gamma[F_\Gamma](s) - \frac{v_\Gamma^{-1}}{s-1}\right),$$

where $q \in C_\Gamma$ is a cusp and $\mathcal{R}_q^\Gamma[F_\Gamma](s)$ is given by

$$\mathcal{R}_q^\Gamma[F_\Gamma](s) = \int_{\mathcal{F}_\Gamma} F_\Gamma(z) E_q^\Gamma(z,s)\mu_{\mathrm{hyp}}(z).$$

We claim that for $\Gamma = \Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, the following identity holds

$$\mathcal{R}_\infty^\Gamma[F_\Gamma](s) = \mathcal{R}_0^\Gamma[F_\Gamma](s).$$

Indeed, if we set α as

$$\alpha = \begin{cases} \left(\begin{smallmatrix} 0 & 1\\ -N & 0\end{smallmatrix}\right), & \Gamma = \Gamma_0(N), \Gamma_1(N);\\ \left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right), & \Gamma = \Gamma(N);\end{cases}$$

then $\alpha^{-1}\Gamma\alpha = \Gamma$ holds. As a result, we have $F_\Gamma(\alpha z) = F_\Gamma(z)$ by virtue of Lemma 4.1.4. Also, we verify that $0 = \alpha^{-1}\infty$: if $\alpha = \left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)$, then using (1.3), we have $0 = \alpha^{-1}\infty$, whereas if $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right)$ then the action of α on $\mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$ coincides with the action of the matrix

$$\sigma_0 = \begin{pmatrix} 0 & 1/\sqrt{N} \\ -\sqrt{N} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$

i.e., the scaling matrix of the cusp 0. Now, if we take $q = \infty$ and $\gamma = \alpha$ in Lemma 1.4.12, then we have

$$E^{\Gamma}_{\infty}(\alpha z, s) = E^{\Gamma}_{0}(z, s).$$

Hence, we have the following identities

$$\begin{aligned} \mathcal{R}^{\Gamma}_{0}[F_{\Gamma}](s) &= \int\limits_{\mathcal{F}_{\Gamma}} F_{\Gamma}(z) E^{\Gamma}_{0}(z, s) \mu_{\mathrm{hyp}}(z) \\ &= \int\limits_{\alpha\mathcal{F}_{\Gamma}} F_{\Gamma}(w) E^{\Gamma}_{\infty}(w, s) \mu_{\mathrm{hyp}}(w) \qquad (4.5) \\ &= \mathcal{R}^{\Gamma}_{\infty}[F_{\Gamma}](s), \end{aligned}$$

where in the last equality we used the fact that $\alpha\mathcal{F}_{\Gamma}$ is a fundamental domain of Γ, provided that $\alpha^{-1}\Gamma\alpha = \Gamma$. This proves the claim and in particular, this yields part (a).

For the proof of part (b), we consider now the matrix

$$\alpha = \begin{pmatrix} x & 1 \\ yN & v' \end{pmatrix}$$

with $x, y \in \mathbb{Z}$ such that $\det(\alpha) = 1$. It can be verified that $\alpha[0:1] = [1:v'] = 0_v$; therefore, we have $0 = \alpha^{-1}0_v$. Since $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and $\alpha \in \Gamma_0(N)$, the identity $\alpha^{-1}\Gamma_1(N)\alpha = \Gamma_1(N)$ holds. Then, by Lemma 4.1.4, we obtain $F_{\Gamma}(\alpha z) = F_{\Gamma}(z)$. In analogy to (4.5), we deduce

$$\mathcal{R}^{\Gamma}_{0_v}[F_{\Gamma}](s) = \mathcal{R}^{\Gamma}_{0}[F_{\Gamma}](s);$$

hence, we have $\mathcal{R}^{\Gamma}_{0_v}[F_{\Gamma}](s) = \mathcal{R}^{\Gamma}_{\infty}[F_{\Gamma}](s)$.

Finally, for the proof of part (c), we take $\alpha = \left(\begin{smallmatrix} -mN & x \\ -v' & 1 \end{smallmatrix}\right)$ with $x, y \in \mathbb{Z}$ such that $\det(\alpha) = 1$, then we have $0_v = \alpha^{-1}\infty$ and $\alpha^{-1}\Gamma(N)\alpha = \Gamma(N)$; hence we have

$$\mathcal{R}^{\Gamma}_{\infty}[F_{\Gamma}](s) = \mathcal{R}^{\Gamma}_{0_v}[F_{\Gamma}](s).$$

This concludes the proof. $\square$

Calculus of the constant $\mathscr{R}^{\Gamma}_{\infty}$

In order to determine the constant $\mathscr{R}^{\Gamma}_{\infty}$, we follow the approach given in [AU97] which, roughly speaking, consist of expressing $\mathscr{R}^{\Gamma}_{\infty}$ as a sum of contributions coming from the elliptic, hyperbolic, and parabolic elements of Γ in the spectral expansions of particular automorphic kernels of weight 0 and 2.

For the following considerations, we write $h(r)$ for the holomorphic continuation of the test function

$$h(r) = e^{-t\left(\frac{1}{4}+r^2\right)} \qquad (t \in \mathbb{R},\, t > 0)$$

(given by (1.17)) to the strip $|\mathrm{Im}(r)| < A/2$ with $A \in \mathbb{R}$, $A > 1$.

First of all, we set functions

$$T_0(z) \coloneqq \sum_{j=1}^{\infty} h(r_j)|\psi_j(z)|^2,$$

$$T_2(z) \coloneqq \sum_{j=1}^{\infty} \frac{h(r_j)}{\lambda_j}|\mathrm{K}_0\psi_j(z)|^2,$$

where ψ_j stands for the eigenfunction of $\widetilde{\Delta}_0$ given by Proposition 1.5.2 with eigenvalue $\lambda_j = 1/4 + r_j$ and K_0 is the Maass operator given by (1.5).

Remark 4.1.5. By [Roe67, Satz 8.1, p. 273] and [Roe66, p. 321], each eigenfunction ψ_j with $j \geq 1$ (that is, ψ_j is non-constant) is in fact an eigenfunction of $\Delta_{\mathrm{hyp},0}$ lying in $\mathscr{S}_{0,\lambda_j}(\Gamma)$. Thus, ψ_j is of rapid decay at every cusp of $\Gamma\backslash\mathbb{H}$.

Now, let us consider the difference

$$\mathcal{T}(z) \coloneqq T_2(z) - T_0(z). \tag{4.6}$$

Lemma 4.1.6. *Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Then the following identity holds*

$$\int_{\mathcal{F}_\Gamma} \mathcal{T}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) = \frac{s(s-1)}{2}\sum_{j=1}^{\infty}\frac{h(r_j)}{\lambda_j}\mathcal{R}_\infty^\Gamma[\,|\psi_j|^2](s).$$

Proof. (Sketch.) For the proof, we first note that if $f \in \mathscr{A}_{0,\lambda}(\Gamma)$ is of rapid decay at ∞, then we have the identity

$$\int_{\mathcal{F}_\Gamma} |\mathrm{K}_0 f(z)|^2 E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) = \left(\lambda + \frac{s(s-1)}{2}\right)\int_{\mathcal{F}_\Gamma} |f(z)|^2 E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z), \tag{4.7}$$

which is an immediate extension of [AU97, Lemme 3.2.18, p. 44] to an arbitrary congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.

Now, observe that

$$\begin{aligned}\int\limits_{\mathcal{F}_\Gamma} \mathcal{T}(z)E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) &= \int\limits_{\mathcal{F}_\Gamma} T_2(z)E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) - \int\limits_{\mathcal{F}_\Gamma} T_0(z)E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z)\\ &= \sum_{j=1}^\infty \frac{h(r_j)}{\lambda_j}\mathcal{R}_\infty^\Gamma[|\mathrm{K}_0\psi_j|^2](s) - \sum_{j=1}^\infty h(r_j)\mathcal{R}_\infty^\Gamma[|\psi_j|^2](s).\end{aligned} \tag{4.8}$$

By taking $f=\psi_j$ and $\lambda=\lambda_j$ in (4.7), the first sum on the right hand side of (4.8) gives

$$\sum_{j=1}^\infty \frac{h(r_j)}{\lambda_j}\mathcal{R}_\infty^\Gamma[|\mathrm{K}_0\psi_j|^2](s) = \sum_{j=1}^\infty \frac{h(r_j)}{\lambda_j}\left(\lambda_j + \frac{s(s-1)}{2}\right)\mathcal{R}_\infty^\Gamma[|\psi_j|^2](s).$$

The result follows by using this identity on the right hand side of (4.8). □

Remark 4.1.7. Recall that the function $\mathcal{R}_\infty^\Gamma[|\psi_j|^2](s)$ possesses a meromorphic continuation with a simple pole at $s=1$ with residue equal to $v_\Gamma^{-1}\langle\psi_j,\psi_j\rangle = v_\Gamma^{-1}$ (see Section 1.6, p. 19); here, $\langle\cdot,\cdot\rangle$ is given by (1.15). Therefore, the right hand side of Lemma 4.1.6 provides the analytic continuation of the left hand side. Furthermore, by the Laurent expansion

$$\mathcal{R}_\infty^\Gamma[|\psi_j|^2](s) = \frac{v_\Gamma^{-1}}{s-1} + \mathscr{R} + O(s-1)$$

at $s=1$, where $\mathscr{R}$ is a constant not depending on s, we obtain

$$\int\limits_{\mathcal{F}_\Gamma} \mathcal{T}(z)E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) = \frac{v_\Gamma^{-1}}{2}\sum_{j=1}^\infty \frac{h(r_j)}{\lambda_j} + O(s-1). \tag{4.9}$$

Secondly, we have the following definition.

Definition 4.1.8. The *elliptic*, *hyperbolic*, and *parabolic parts of the automorphic kernel* $\mathcal{K}_k^\Gamma(z,w)$ of weight k defined in Definition 1.5.5 are given by

$$\mathcal{K}_k^{\mathrm{ell}}(z,w) \coloneqq \frac{1}{2}\sum_{\substack{\gamma\in\{\pm I\}\Gamma\\ |\mathrm{tr}(\gamma)|<2}} j_\gamma(w;k)\pi_k(z,\gamma w);$$

$$\mathcal{K}_k^{\mathrm{hyp}}(z,w) \coloneqq \frac{1}{2}\sum_{\substack{\gamma\in\{\pm I\}\Gamma\\ |\mathrm{tr}(\gamma)|>2}} j_\gamma(w;k)\pi_k(z,\gamma w);$$

$$\mathcal{K}_k^{\mathrm{par}}(z,w) \coloneqq \frac{1}{2}\left(\sum_{\substack{\gamma\in\{\pm I\}\Gamma\\ |\mathrm{tr}(\gamma)|=2}} j_\gamma(w;k)\pi_k(z,\gamma w)\right) - S_k(z,w);$$

respectively, where $\pi_k(z,w)$ is the point-pair invariant associated to $h(r)$ and $S_k(z,w)$ is given by

$$S_k(z,w) := \frac{1}{4\pi}\sum_{q\in C_\Gamma}\int\limits_{-\infty}^{\infty} h(r)E^{\Gamma}_{q,k}\Big(z,\frac{1}{2}+ir\Big)\overline{E^{\Gamma}_{q,k}\Big(w,\frac{1}{2}+ir\Big)}\mathrm{d}r + \frac{2-k}{2}v_{\Gamma}^{-1}.$$

Notation 4.1.9. In the sequel, we write $\mathcal{K}_k^{\mathrm{ell}}(z)$, $\mathcal{K}_k^{\mathrm{hyp}}(z)$, $\mathcal{K}_k^{\mathrm{par}}(z)$, $S_k(z)$, and $\mathcal{K}_k^{\Gamma}(z)$ to denote $\mathcal{K}_k^{\mathrm{ell}}(z,z)$, $\mathcal{K}_k^{\mathrm{hyp}}(z,z)$, $\mathcal{K}_k^{\mathrm{par}}(z,z)$, $S_k(z,z)$, and $\mathcal{K}_k^{\Gamma}(z,z)$ respectively, where $\mathcal{K}_k^{\Gamma}(z,w)$ is the automorphic kernel of weight k given in Definition 1.5.5.

Now, we set

$$\nu_k(\gamma;z) := j_\gamma(z;k)\pi_k(z,\gamma z). \tag{4.10}$$

Remark 4.1.10. By [Hej76, Proposition 2.11, p. 359] and the multiplicative property of $j_\gamma(z;k)$ (see Section 1.3 of Chapter 1), we obtain

$$\nu_k(\gamma;\alpha z) = \nu_k(\alpha^{-1}\gamma\alpha;z).$$

Using this identity, a direct calculation shows that $\mathcal{K}_k^{\mathrm{ell}}(z)$, $\mathcal{K}_k^{\mathrm{hyp}}(z)$, and $\mathcal{K}_k^{\mathrm{par}}(z)$ belong to $\mathscr{F}_0(\Gamma)$.

Lemma 4.1.11. *Let $s\in\mathbb{C}$ with $1<\mathrm{Re}(s)<1+A$ and $k=0,2$. Then the integrals*

$$\int\limits_{\mathcal{F}_\Gamma}\mathcal{K}_k^{\mathrm{ell}}(z)E^{\Gamma}_{\infty}(z,s)\mu_{\mathrm{hyp}}(z),$$

$$\int\limits_{\mathcal{F}_\Gamma}\mathcal{K}_k^{\mathrm{hyp}}(z)E^{\Gamma}_{\infty}(z,s)\mu_{\mathrm{hyp}}(z),$$

exist.

Proof. The proof follows from extended versions of the results given by [AU97, Proposition 3.2.1, p. 26] and [May12, Lemma 4.4.1, p. 94]. ☐

Thirdly, let us consider the following functions

$$\mathcal{E}(z) := \mathcal{K}_2^{\mathrm{ell}}(z) - \mathcal{K}_0^{\mathrm{ell}}(z);$$

$$\mathcal{H}(z) := \mathcal{K}_2^{\mathrm{hyp}}(z) - \mathcal{K}_0^{\mathrm{hyp}}(z);$$

$$\mathcal{P}(z) := \mathcal{K}_2^{\mathrm{par}}(z) - \mathcal{K}_0^{\mathrm{par}}(z).$$

Hence, we have

$$\mathcal{K}_2^\Gamma(z) - \mathcal{K}_0^\Gamma(z) = \mathcal{E}(z) + \mathcal{H}(z) + \mathcal{P}(z) + S_2(z) - S_0(z). \tag{4.11}$$

From the spectral decomposition of automorphic kernels given in Theorem 1.5.7, the difference $\mathcal{K}_2^\Gamma(z) - \mathcal{K}_0^\Gamma(z)$ is given by

$$\begin{aligned}\mathcal{K}_2^\Gamma(z) - \mathcal{K}_0^\Gamma(z) = \sum_{j=1}^{g_\Gamma} \operatorname{Im}(z)^2 |f_j(z)|^2 + \sum_{j=1}^{\infty} \frac{h(r_j)}{\lambda_j} |(\mathrm{K}_0\psi_j)(z)|^2 + S_2(z)\\ - \sum_{j=1}^{\infty} h(r_j)|\psi_j(z)|^2 - S_0(z).\end{aligned}$$

Using

$$\sum_{j=1}^{g_\Gamma} \operatorname{Im}(z)^2 |f_j(z)|^2 = g_{\scriptscriptstyle\Gamma} F_\Gamma(z)$$

together with (4.6) and (4.11) we deduce the identity

$$g_{\scriptscriptstyle\Gamma} F_\Gamma(z) = -\mathcal{T}(z) + \mathcal{E}(z) + \mathcal{H}(z) + \mathcal{P}(z).$$

By taking the integral over a fundamental domain $\mathcal{F}_\Gamma$ against the Eisenstein series $E_\infty^\Gamma(z,s)$ on both sides of the previous equality, we obtain

$$\begin{aligned} g_{\scriptscriptstyle\Gamma} \mathcal{R}_\infty^\Gamma[F_\Gamma](s) = -\int_{\mathcal{F}_\Gamma} \mathcal{T}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) + \int_{\mathcal{F}_\Gamma} \mathcal{E}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z)\\ + \int_{\mathcal{F}_\Gamma} \mathcal{H}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) + \int_{\mathcal{F}_\Gamma} \mathcal{P}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z).\end{aligned} \tag{4.12}$$

This equality holds for $s \in \mathbb{C}$ with $1 < \operatorname{Re}(s) < A$; however, by Lemma 4.1.6, Corollary 4.3.4, Corollary 4.4.10, and [AU97, Proposition 3.3.5, p. 58]; each integral on the right hand side of (4.12) can be analytically continued to the half-plane $\operatorname{Re}(s) < A$.

Notation 4.1.12. Let us denote by $\mathscr{R}_\infty^{\mathrm{ell}}$, $\mathscr{R}_\infty^{\mathrm{hyp}}$, and $\mathscr{R}_\infty^{\mathrm{par}}$ the constant terms of the expansion at $s = 1$ of the (analytically continued) integrals

$$\int_{\mathcal{F}_\Gamma} \mathcal{E}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z), \int_{\mathcal{F}_\Gamma} \mathcal{H}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z), \text{ and } \int_{\mathcal{F}_\Gamma} \mathcal{P}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z)$$

respectively.

Observe that

$$g_{\scriptscriptstyle\Gamma} \mathcal{R}_\infty^\Gamma[F_\Gamma](s) = \frac{g_{\scriptscriptstyle\Gamma} v_{\scriptscriptstyle\Gamma}^{-1}}{s-1} + g_{\scriptscriptstyle\Gamma} \mathscr{R}_\infty^\Gamma + O(s-1), \tag{4.13}$$

by Section 1.6, p. 19. Then by comparing the Laurent expansions at $s = 1$ on both sides of (4.12), we obtain using (4.9) and (4.13) the following identity

$$g_{\Gamma}\mathscr{R}_{\infty}^{\Gamma} = -\frac{v_{\Gamma}^{-1}}{2}\sum_{j=1}^{\infty}\frac{h(r_j)}{\lambda_j} + \mathscr{R}_{\infty}^{\mathrm{ell}} + \mathscr{R}_{\infty}^{\mathrm{hyp}} + \mathscr{R}_{\infty}^{\mathrm{par}}. \tag{4.14}$$

Definition 4.1.13. The constants $\mathscr{R}_{\infty}^{\mathrm{ell}}$, $\mathscr{R}_{\infty}^{\mathrm{hyp}}$, and $\mathscr{R}_{\infty}^{\mathrm{par}}$ given in (4.14) and defined in Notation 4.1.12 are called the *elliptic*, *hyperbolic*, and *parabolic contribution*, respectively.

In the next propositions we state the final form of these contributions in terms of the level N.

Proposition 4.1.14. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 5$ is an odd square-free integer such that $g_{\Gamma} > 1$. Then the following assertions hold:*

(a) If Γ is either $\Gamma_1(N)$ or $\Gamma(N)$, then we have $\mathscr{R}_{\infty}^{\mathrm{ell}} = 0$.

(b) If $\Gamma = \Gamma_0(N)$, then we have

$$\mathscr{R}_{\infty}^{\mathrm{ell}} = \left(\prod_{p|N}\left(1 + \left(\frac{-1}{p}\right)\right)\right)\left[\frac{1}{4}\sum_{p|N}\frac{p}{1+p}\log(p) + C_0\right]$$
$$+ \left(\prod_{p|N}\left(1 + \left(\frac{-3}{p}\right)\right)\right)\left[\frac{1}{3}\sum_{p|N}\frac{p}{1+p}\log(p) + C_1\right],$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol and C_0, C_1 are real constants.

Proof. For the proof of the first assertion, note that there are no elliptic elements in $\Gamma_1(N)$ and $\Gamma(N)$ for $N \geq 5$. For the second assertion, we refer the reader to [AU97, Proposition 3.3.5, p. 58]. □

Proposition 4.1.15. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_{\Gamma} > 1$. Then the following identity holds*

$$\mathscr{R}_{\infty}^{\mathrm{hyp}} = -\frac{v_{\Gamma}^{-1}}{2}\left(\int_0^t \Big(\Theta_{\Gamma}(u) - 1\Big)\mathrm{d}u + t\right),$$

where t is the fixed positive real in the definition of the function $h(r)$, $\Theta_{\Gamma}(u)$ is given by

$$\Theta_{\Gamma}(u) = \sum_{\substack{l\in\mathbb{Z}\\ |l|>2}}\sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma}\frac{\log(N(\gamma_0))}{\sqrt{l^2-4}}g_u(2\log(\eta_l))$$

with $g_u(v) = (1/\sqrt{4\pi u})e^{-(v^2/u+u)/4}$ *and* $\eta_l = (l + \sqrt{l^2-4})/2$.

Proposition 4.1.16. *Let* Γ *be one of the congruence subgroups* $\Gamma_0(N)$, $\Gamma_1(N)$, *or* $\Gamma(N)$, *where* $N \geq 3$ *is an odd square-free integer such that* $g_\Gamma > 1$. *Then the following assertions hold:*

(a) If $\Gamma = \Gamma_0(N)$, *then we have*

$$\begin{aligned}\mathscr{R}_\infty^{\mathrm{par}} &= \frac{1-\log(4\pi)}{4\pi} + \frac{\mathscr{C}_{\infty\infty}^{\Gamma}}{2} + \frac{v_\Gamma^{-1}}{2}(t+1) + \gamma_{\mathrm{EM}} C_2(t) + C_3(t) + C_4(t) \\ &\quad + \frac{d(N)}{v_\Gamma}\bigg[\mathcal{C}(t) - A_2(t)\bigg(\sum_{p|N}\frac{1+2p}{1+p}\log(p) - \frac{\varphi(N)}{Nd(N)} - \frac{1}{2}\log(N)\bigg)\bigg].\end{aligned}$$

(b) If $\Gamma = \Gamma_1(N)$, *then we have*

$$\begin{aligned}\mathscr{R}_\infty^{\mathrm{par}} &= \frac{1-\log(4\pi)}{4\pi} + \frac{\mathscr{C}_{\infty\infty}^{\Gamma}}{2} + \frac{v_\Gamma^{-1}}{2}(t+1) + \gamma_{\mathrm{EM}} C_2(t) + C_3(t) + C_4(t) \\ &\quad + \frac{\varphi(N)}{v_\Gamma}\prod_{p|N}\Big(1+\frac{2}{p}\Big)\bigg[\mathcal{C}(t) - A_2(t)\bigg(\sum_{p|N}\frac{1+2p}{1+p}\log(p) \\ &\quad - \prod_{p|N}\Big(\frac{1}{p+2}\Big)\Big(1+\sum_{j=1}^{\omega(N)} 2^{\omega(N)-j}\sum_{\substack{d|N\\ \omega(d)=j}} d\log(d)\Big)\bigg)\bigg].\end{aligned}$$

(c) If $\Gamma = \Gamma(N)$, *then we have*

$$\begin{aligned}\mathscr{R}_\infty^{\mathrm{par}} &= \frac{1-\log(4\pi)}{4\pi} + \frac{\mathscr{C}_{\infty\infty}^{\Gamma}}{2} + \frac{v_\Gamma^{-1}}{2}(t+1) + \gamma_{\mathrm{EM}} C_2(t) + C_3(t) + C_4(t) \\ &\quad + 2\frac{N\varphi(N)}{v_\Gamma}\prod_{p|N}\Big(1+\frac{1}{p}\Big)\Big[\mathcal{C}(t) - 2A_2(t)\log(N)\Big].\end{aligned}$$

Here, t *is the fixed positive real in the definition of the function* $h(r)$. *Furthermore,* $C_j(t)$ $(j = 1,2,3,4)$ *depends only on* t *and has limit equal to* $C_j \in \mathbb{R}$ *as* $t \to \infty$, *where* C_1, C_2 *are the real constants given in Proposition 4.1.14 and* $\mathcal{C}(t)$ *is equal to* $C_1(t) + A_2(t)\big(2\mathscr{C} + \gamma_{\mathrm{EM}}\big)$.

The proofs of propositions 4.1.15 and 4.1.16 will be presented in sections 4.3 and 4.4, respectively.

4.2 The hyperbolic contribution. I

This subsection is a brief digression on zeta functions associated to matrices of congruence subgroups with fixed trace. In particular, we will be interested in their residue at $s = 1$. Such functions will appear naturally in the proof of Proposition 4.1.15.

For the following considerations, let l be a fixed integer such that $|l| > 2$ and denote by $\mathcal{M}_l(\mathbb{Z})$ the set of matrices in $\mathrm{SL}_2(\mathbb{Z})$ having trace equal to l. We set $\Delta := l^2 - 4$ and for a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, we put $\mathrm{sp}_l(\Gamma) := (\{\pm I\}\Gamma) \cap \mathcal{M}_l(\mathbb{Z})$.

Definition 4.2.1. A *binary quadratic form* is a homogeneous polynomial in two variables of degree 2, i.e., $f(x, y) = ax^2 + bxy + cy^2$, where $a, b, c \in \mathbb{Z}$ and x, y are variables.

Notation 4.2.2. In the sequel we will also denote the binary quadratic form $f(x, y) = ax^2+bxy+cy^2$ by $f = [a, b, c]$. The integer $\Delta_f := b^2-4ac$ is called the *discriminant of* $f(x, y)$. We denote by $\mathrm{BQF}(\Delta)$ the set of all binary quadratic forms of fixed discriminant Δ.

The following two facts can be easily verified:

(i) The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathrm{BQF}(\Delta)$ as follows: given $\gamma = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $f = [a, b, c] \in \mathrm{BQF}(\Delta)$, the action is given by

$$f \circ \gamma := [a', b', c'],$$

where a', b', c' are given by

$$\begin{aligned} a' &= ax^2 + bxz + cz^2; \\ b' &= b(xt + yz) + 2(axy + czt); \\ c' &= ay^2 + byt + ct^2 \end{aligned}$$

(see [Zag81b, p. 58]).

(ii) The map $\rho : \mathcal{M}_l(\mathbb{Z}) \longrightarrow \mathrm{BQF}(\Delta)$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto [c, d - a, -b]$, yields a bijection with inverse $[A, B, C] \longmapsto \left(\begin{smallmatrix} (l-B)/2 & -C \\ A & (l-B)/2 \end{smallmatrix}\right)$ (see [Zag77, §3, p. 124]).

In what follows, we set $f_\gamma := \rho(\gamma)$, $\gamma_f := \rho^{-1}(f)$ and $Q_l(\Gamma) := \rho(\mathrm{sp}_l(\Gamma))$. Note that with this notation, we have $f_{\gamma_f} = f$ and $\gamma_{f_\gamma} = \gamma$.

Lemma 4.2.3. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Then the assignment $(\delta, f) \longmapsto f \circ \delta$ gives a well-defined action of Γ on $Q_l(\Gamma)$.*

Proof. For the proof, let $\delta \in \Gamma$ and $f \in Q_l(\Gamma)$. Then, it can be verified that

$$f \circ \delta = f_{\gamma_f} \circ \delta = f_{\delta^{-1}\gamma_f\delta},$$

where $\gamma_f \in \mathrm{sp}_l(\Gamma)$. Since $\delta^{-1}\gamma_f\delta \in \mathrm{sp}_l(\Gamma)$, we have $f \circ \delta = f_{\delta^{-1}\gamma_f\delta} \in Q_l(\Gamma)$. This concludes the proof. $\square$

Notation 4.2.4. In the sequel, we let $M_\Gamma \subset \mathbb{Z}^2$ be a subset of $\mathbb{Z}^2$ which is Γ-invariant under right multiplication, that is, if $(m,n) \in M_\Gamma$, then the pair $(m,n)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (am+cn, bm+dn) \in M_\Gamma$. Let us write

$$f \cdot (x,y) \coloneqq f(y,-x)$$

as in [Zag77, p. 131] and for a given $f \in \mathrm{BQF}(\Delta)$, we define the set

$$M_\Gamma^f \coloneqq \{(m,n) \in M_\Gamma \mid f \cdot (m,n) > 0\}.$$

Lemma 4.2.5. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Then the set*

$$S_\Gamma = \{(f,(m,n)) \mid (m,n) \in M_\Gamma^f \text{ with } f \in Q_l(\Gamma)\} \subset Q_l(\Gamma) \times M_\Gamma$$

is invariant under the simultaneous action of Γ in each coordinate.

Proof. Suppose that $(f,(m,n)) \in S_\Gamma$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$; we have to show that $(f \circ \gamma, (m,n)\gamma) \in S_\Gamma$, i.e., $(m,n)\gamma \in M_\Gamma^{f\circ\gamma}$. For this, we claim that the identity

$$(f \circ \gamma) \cdot ((m,n)\gamma) = f \cdot (m,n) \tag{4.15}$$

holds. Indeed

$$\begin{aligned}(f \circ \gamma) \cdot ((m,n)\gamma) &= (f \circ \gamma) \cdot (am+cn, bm+dn) \\ &= (f \circ \gamma)(bm+dn, -(am+cn)) \\ &= f\Big((bm+dn, -am-cn)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^t\Big) \\ &= f(n(ad-bc), -m(ad-bc)) = f(n,-m),\end{aligned}$$

where in the third equality we used $(f \circ \gamma)(x,y) = f((x,y)\gamma^t)$ with γ^t the transpose of γ. Since $(f,(m,n)) \in S_\Gamma$, we have in particular that $f \cdot (m,n) > 0$; therefore, the previous claim implies $(f \circ \gamma) \cdot ((m,n)\gamma) > 0$. This concludes the proof. $\square$

Note that, by virtue of Lemma 4.2.5, the subgroup Γ acts on elements of S_Γ coordinate-wise. In the sequel, we consider only this action on the set S_Γ.

Lemma 4.2.6. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and S_Γ the set given* *in Lemma 4.2.5. Then we have*

$$S_\Gamma/\Gamma \simeq \bigsqcup_{f\in Q_l(\Gamma)/\Gamma} \Big(M_\Gamma^f/\mathrm{Stab}_\Gamma(f)\Big),$$

where $\mathrm{Stab}_\Gamma(f)$ *denotes the stabilizer of* f *under the action of* Γ *on* $Q_l(\Gamma)$*.*

Proof. This is a consequence of the "allgemeine Prinzip" in [Zag81b, p. 66] with $G = \Gamma$, $X = Q_l(\Gamma)$, and $Y = M_\Gamma$. In this case, $Y_f = M_\Gamma^f$ for each $f \in Q_l(\Gamma)$. $\square$

Definition 4.2.7. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, $M_\Gamma \subset \mathbb{Z}^2$ a given Γ-invariant subset under right multiplication, $\gamma \in \mathrm{sp}_l(\Gamma)$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The *zeta function associated to* γ is defined by

$$\zeta_\gamma(s) := \sum_{(m,n)\in M_\Gamma^{f_\gamma}/\mathrm{Stab}_\Gamma(f_\gamma)} f_\gamma(n, -m)^{-s}.$$

To illustrate the previous notions, let us consider the following prototypical example.

Example 4.2.8. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $M_\Gamma = \mathbb{Z}^2 \setminus \{(0,0)\}$ with $l \in \mathbb{Z}$ an integer satisfying $|l| > 2$. Consider $\gamma \in \mathrm{sp}_l(\Gamma)$ of the form

$$\gamma = \begin{pmatrix} \frac{l-b}{2} & -c \\ a & \frac{l+b}{2} \end{pmatrix},$$

where a, b, c are integers such that $\gcd(a, b, c) = 1$, the corresponding binary quadratic form is $f_\gamma = [a, b, c]$. Let $D = b^2 - 4ac = l^2 - 4$, $L = \mathbb{Q}(\sqrt{D})$, and $\mathcal{O}_L$ the ring of integers of L. Further, for $\theta = (b + \sqrt{D})/2a$, set

$$\mathfrak{a}_{f_\gamma} = \mathbb{Z} + \mathbb{Z}\theta,$$

and set

$$U_+ = \{\alpha \in \mathcal{O}_L^\times \mid N(\alpha) = 1\},$$

where $N(\alpha)$ denotes the norm of the element $\alpha \in L$. We emphasize the following two facts:

(i) The map $M_\Gamma^{f_\gamma} \longrightarrow \{\xi \in \mathfrak{a}_{f_\gamma} \mid N(\xi) > 0\}$ given by $(m, n) \longmapsto m + n\theta$ descents to a bijection

$$M_\Gamma^{f_\gamma}/\mathrm{Stab}_\Gamma(f_\gamma) \;\simeq\; \{\xi \in \mathfrak{a}_{f_\gamma} \mid N(\xi) > 0\}/U_+$$

(see [Zag81b, §11, pp. 96]).

(ii) The map $\{\xi \in \mathfrak{a}_{f_\gamma} \mid N(\xi) > 0\}/U_+ \longrightarrow \{\mathfrak{b} \in \mathcal{A}^{-1} \mid \mathfrak{b} \subset \mathcal{O}_L\}$ given by $\xi U_+ \longmapsto (\xi)\mathfrak{a}_{f_\gamma}^{-1}$ defines a bijection (see [Zag81b, p. 98]); here, (ξ) is the principal ideal generated by ξ and $\mathcal{A}$ is the narrow ideal class of $\mathfrak{a}_{f_\gamma}$ (see [Zag81b, Definition, p. 91]).

Let $\mathcal{N}$ denote the ideal norm. If $\xi = n - m\theta \in \mathfrak{a}_{f_\gamma}$ such that $N(\xi) > 0$, then $f_\gamma(n, -m) = \mathcal{N}\Big((\xi)\mathfrak{a}_{f_\gamma}^{-1}\Big)$. Indeed, by [Zag81b, (5), p. 89] and [Zag81b, p. 93], we have $\mathcal{N}\Big((\xi)\Big) = |N(\xi)| = N(\xi)$ and $\mathcal{N}\mathfrak{a}_{f_\gamma} = 1/a$, respectively. Since we have

$$\begin{aligned} N(\xi) &= (n - m\theta)(n - m\overline{\theta}) \\ &= \left(n - m\left(\frac{b + \sqrt{D}}{2a}\right)\right)\left(n - m\left(\frac{b - \sqrt{D}}{2a}\right)\right) \\ &= n^2 + \frac{b}{a}n(-m) + \frac{c}{a}m^2, \end{aligned}$$

the assertion follows. Therefore, fact (i) implies that

$$\zeta_\gamma(s) = \sum_{(m,n) \in M_\Gamma^{f_\gamma}/\mathrm{Stab}_\Gamma(f_\gamma)} f_\gamma(n, -m)^{-s} = \sum_{\substack{\xi \in \mathfrak{a}_{f_\gamma}/U_+ \\ N(\xi) > 0}} \mathcal{N}\Big((\xi)\mathfrak{a}_{f_\gamma}^{-1}\Big)^{-s}.$$

Taking into account fact (ii), we then obtain

$$\zeta_\gamma(s) = \sum_{\substack{\mathfrak{b} \in \mathcal{A}^{-1} \\ 0 \neq \mathfrak{b} \subset \mathcal{O}_L}} \mathcal{N}\mathfrak{b}^{-s} = \zeta(s, \mathcal{A}^{-1}) = \zeta(s, \mathcal{A}),$$

where $\zeta(s, \mathcal{A})$ is the zeta function associated to the narrow ideal class $\mathcal{A}$ given by

$$\zeta(s, \mathcal{A}) = \sum_{\substack{\mathfrak{c} \in \mathcal{A} \\ 0 \neq \mathfrak{c} \subset \mathcal{O}_L}} \frac{1}{(\mathcal{N}\mathfrak{c})^s}$$

(see Appendix C.2). Note that the identity $\zeta_\gamma(s) = \zeta(s, \mathcal{A})$ provides the meromorphic continuation of $\zeta_\gamma(s)$ to the half-plane $\mathrm{Re}(s) > 1/2$. Moreover, it gives the residue at $s = 1$, namely, observing $D > 0$, we have

$$\mathrm{res}_{s=1}\Big(\zeta_\gamma(s)\Big) = \frac{\log(\varepsilon_\gamma)}{\sqrt{D}},$$

where ε_γ is a generator of U_+ (see [Zag81b, Satz 2, p. 103]).

The previous example showed that for the modular group, the zeta function associated to $\gamma \in \mathrm{sp}_l(\mathrm{SL}_2(\mathbb{Z}))$ with $M_\Gamma = \mathbb{Z}^2 \setminus \{(0,0)\}$ is the zeta function of certain ideal class of a real quadratic field. In the general case of congruence subgroups with level N, our approach uses partial zeta functions associated to ideals in the ray class group of a real quadratic field.

For the rest of this section, we will use the terminology and notations of Appendix C which is a summary of ray class groups and partial zeta functions.

Let L be a quadratic field with ring of integers $\mathcal{O}_L$ and suppose that $\mathfrak{f} \subset \mathcal{O}_L$ is a nonzero integral ideal. In what follows, we will consider cycles of L of the form

$$\mathfrak{m} = \mathfrak{m}_\infty \mathfrak{f} \tag{4.16}$$

with $m(v) > 0$, for all archimedean places $v|\mathfrak{m}_\infty$. Let $U_\mathfrak{m}$ be the subgroup of $\mathcal{O}_L^\times$ consisting of those units ξ satisfying $\xi \equiv 1 \bmod^* \mathfrak{m}$ (see Definition C.1.4).

Notation 4.2.9. For a given subset $X \subset L$, we denote by X_+ the set of all totally positive elements of X (see Appendix C.1).

Lemma 4.2.10. *Let L be a real quadratic field. Suppose that $\mathfrak{C} \in Cl_\mathfrak{m}(L)$ is an $\mathfrak{m}$-ideal class with $\mathfrak{m}$ given by (4.16) and fix a nonzero integral ideal $\mathfrak{a} \in \mathfrak{C}$. Let $\mathfrak{b} = \mathbb{Z} + \mathbb{Z}\theta$ ($\theta \in \mathbb{R}$, $\theta > 1$) be a fractional ideal in the narrow ideal class of $\mathfrak{a}^{-1}\mathfrak{f}$ and $z \in L^\times$ with $N(z) > 0$ such that $\mathfrak{b} = (z)\mathfrak{a}^{-1}\mathfrak{f}$. Then the partial zeta function $\zeta(s, \mathfrak{C})$ (see Definition C.2.1) satisfies the identity*

$$\zeta(s, \mathfrak{C}) = (\mathcal{N}\mathfrak{f})^{-s} \sum_{\beta \in (z+\mathfrak{b})_+/U_\mathfrak{m}} \left(\frac{\mathcal{N}(\beta)}{\mathcal{N}\mathfrak{b}} \right)^{-s},$$

where $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1/2$, and $U_\mathfrak{m} = (\mathcal{O}_L^\times \cap (1 + \mathfrak{f}))_+$.

Proof. For the proof we refer the reader to [Yam08, p. 430]. □

Let $u \in (\mathbb{Z}/N\mathbb{Z})^\times$. For the purpose of study the hyperbolic contribution of the congruence subgroup $\Gamma(N)$, we define the subset

$$M_{\Gamma(N)}(u) := \{(m, n) \in \mathbb{Z}^2 \mid (m, n) \equiv (0, u) \bmod N\} \tag{4.17}$$

which is $\Gamma(N)$-invariant under right multiplication.

Proposition 4.2.11. *Let $\Gamma = \Gamma(N)$ with $N \geq 3$ odd and square-free and $M_{\Gamma(N)}(u)$ the set given by (4.17) for some $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ previously chosen. Suppose that $l \in \mathbb{Z}$ is an odd integer satisfying $|l| > 2$ and $l \equiv 2 \bmod N$. Let $\gamma \in \mathrm{sp}_l(\Gamma)$ be a matrix of the form*

$$\gamma = \begin{pmatrix} \frac{l-bN}{2} & -cN \\ aN & \frac{l+bN}{2} \end{pmatrix}$$

with $a, b, c \in \mathbb{Z}$ such that $a \neq 0$, $\gcd(a, b, c) = 1$, and $D = b^2 - 4ac$ is not a *square. Then the following identity holds*

$$\zeta_\gamma(s) = \left(\frac{1}{N}\right)^s \zeta(s, \mathfrak{C})$$

for some $\mathfrak{m}$*-ideal class* $\mathfrak{C} \in Cl_{\mathfrak{m}}(L)$ *and* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1$*; here,* $L = \mathbb{Q}(\sqrt{D})$ *and* $\mathfrak{m}$ *is given by* (4.16) *with* $\mathfrak{f} = (N) \subset \mathcal{O}_L$*. Furthermore, we have*

$$\mathrm{res}_{s=1}\Big(\zeta_\gamma(s)\Big) = \frac{\log(\varepsilon_\gamma)}{N^3\sqrt{D}}\,,$$

where ε_γ *is a generator of* $(\mathcal{O}_L^\times \cap (1+\mathfrak{f}))_+$*.*

Proof. For the proof, observe that the corresponding binary quadratic form associated to γ is $f_\gamma = Ng_\gamma$, where $g_\gamma = [a, b, c]$. Note that $l^2 - 4 = \Delta_{f_\gamma} = N^2D$. Since l, N are odd, we have $D \equiv 1 \bmod N$ which in turn implies that b is odd.

First of all, we claim that

$$\mathrm{Stab}_{\Gamma(N)}(f_\gamma) = \left\{ \begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix} \;\middle|\; t^2 - Du^2 = 4,\ u \equiv 0 \bmod N, \quad \frac{t-u}{2} \equiv 1 \bmod N \right\}. \tag{4.18}$$

Indeed, since $\mathrm{Stab}_{\Gamma(N)}(f_\gamma) = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(g_\gamma) \cap \Gamma(N)$, every $\delta \in \mathrm{Stab}_{\Gamma(N)}(f_\gamma)$ has the form

$$\delta = \begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix}$$

with t, u satisfy $t^2 - Du^2 = 4$. On the one hand, suppose that $d = \gcd(u, N)$, then a direct calculation shows that $a \equiv b \equiv c \equiv 0 \bmod (N/d)$. This would contradict the condition $\gcd(a, b, c) = 1$, unless $d = N$. Therefore, $u \equiv 0 \bmod N$. On the other hand, it is easy to note that

$$\frac{t-bu}{2} \equiv 1 \bmod N \Longleftrightarrow \frac{t-u}{2} \equiv 1 \bmod N.$$

This proves the claim.

Secondly, let $L = \mathbb{Q}(\sqrt{D})$ and consider the cycle $\mathfrak{m}$ given by (4.16) with $\mathfrak{f} = (N) \subset \mathcal{O}_L$. We claim that the map

$$\begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix} \longmapsto \frac{t+u\sqrt{D}}{2}$$

establishes a bijection

$$\mathrm{Stab}_{\Gamma(N)}(f_\gamma) \simeq (\mathcal{O}_L^\times \cap (1+\mathfrak{f}))_+. \tag{4.19}$$

For the injectivity and surjectivity, we refer the reader to [Zag81b, p. 65]. We proceed to show the well-definition of the map. Suppose that

$$\alpha = \begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix} \in \mathrm{Stab}_{\Gamma(N)}(f_\gamma),$$

i.e., we have $t^2 - Du^2 = 4$, $u \equiv 0 \bmod N$, and $(t-u)/2 \equiv 1 \bmod N$ because of (4.18). Then we need to show that the real number $\varepsilon_\alpha := (t + u\sqrt{D})/2$ satisfies: (i) $\varepsilon_\alpha \in 1 + \mathfrak{f}$ and (ii) $N(\varepsilon) > 0$. For the proof of (i), note that since $D \equiv 1$ mod N, then we have

$$\mathcal{O}_L = \mathbb{Z} + \mathbb{Z}\,\frac{1+\sqrt{D}}{2}$$

(see [Zag81b, (1), p. 87]). Therefore, if we write

$$\varepsilon_\alpha = \frac{t+u\sqrt{D}}{2} = \frac{t-u}{2} + u\left(\frac{1+\sqrt{D}}{2}\right),$$

then the condition $(t-u)/2 \equiv 1$ mod N gives

$$\varepsilon_\alpha = 1 + xN + u\left(\frac{1+\sqrt{D}}{2}\right)$$

for some $x \in \mathbb{Z}$, whereas the condition $u \equiv 0$ mod N gives

$$\varepsilon_\alpha = 1 + N\left(x + y\left(\frac{1+\sqrt{D}}{2}\right)\right)$$

for some $y \in \mathbb{Z}$. As a result, $\varepsilon_\alpha \in 1 + (N)$. For the proof of (ii), note that

$$N(\varepsilon_\alpha) = \frac{t^2 - Du^2}{4} = 1.$$

This proves the claim.

Thirdly, note that by the identity (4.15), the stabilizer $\mathrm{Stab}_{\Gamma(N)}(f_\gamma)$ acts on $M^{f_\gamma}_{\Gamma(N)}(u)$ by right multiplication. Now, set $\theta = -(b+\sqrt{D})/2c$ and $\mathfrak{b} = \mathbb{Z} + \mathbb{Z}\theta$, and consider the action of the totally positive units on $(u/N + \mathfrak{b})_+$ given by $(\varepsilon, u/N + \xi) \longmapsto u/N + \varepsilon\xi$. Then the natural map $M^{f_\gamma}_{\Gamma(N)}(u) \longrightarrow (u/N + \mathfrak{b})_+$ defined by $(n, -m) \longmapsto u/N + n - m\theta$ is compatible under the previous actions, namely, we have

$$(n, -m)\begin{pmatrix} \frac{t-bv}{2} & -cv \\ av & \frac{t+bv}{2} \end{pmatrix} = \varepsilon(n + (-m)\theta),$$

where $\varepsilon = (t + v\sqrt{D})/2$. Consequently, by virtue of (4.19), we have

$$M^{f_\gamma}_{\Gamma(N)}(u)/\mathrm{Stab}_{\Gamma(N)}(f_\gamma) \;\simeq\; (u/N + \mathfrak{b})_+/(\mathcal{O}_L^\times \cap (1+\mathfrak{f}))_+.$$

Fourthly, if $\xi = (u/N) + n + (-m)\theta$ is an element of $(u/N + \mathfrak{b})_+$, then we have

$$f_\gamma(u + nN, -Nm) = Ng_\gamma(u + nN, -Nm)$$

$$= N^3\left[a\Big(\frac{u}{N}+n\Big)^2 + b\Big(\frac{u}{N}+n\Big)(-m) + cm^2\right]$$
$$= N^3\frac{N(\xi)}{\mathcal{N}\mathfrak{b}};$$

therefore, we obtain

$$\zeta_\gamma(s) = \sum_{(m,n)\in M^{f_\gamma}_{\Gamma(N)}(u)/\mathrm{Stab}_{\Gamma(N)}(f_\gamma)} f_\gamma(n,-m)^{-s}$$
$$= N^{-s}\sum_{\xi\in((u/N)+\mathfrak{b})_+/U_\mathfrak{m}} \mathcal{N}\Big((\xi)\mathfrak{b}^{-1}\mathfrak{f}\Big)^{-s},$$

where $\mathfrak{f}=(N)$ and $U_\mathfrak{m} = (\mathcal{O}_L^\times \cap (1+\mathfrak{f}))_+$. Furthermore, by Lemma 4.2.10, we obtain

$$\sum_{\xi\in((u/N)+\mathfrak{b})_+/U_\mathfrak{m}} \mathcal{N}\Big((\xi)\mathfrak{b}^{-1}\mathfrak{f}\Big)^{-s} = \sum_{\substack{\mathfrak{c}\in\mathfrak{C}\\ 0\neq\mathfrak{c}\subset\mathcal{O}_L}} (\mathcal{N}\mathfrak{c})^{-s},$$

where $\mathfrak{C}$ is the $\mathfrak{m}$-ideal class of $\mathfrak{a} = \mathfrak{f}\mathfrak{b}^{-1}(u/N)$. Consequently, we have

$$\zeta_\gamma(s) = \Big(\frac{1}{N}\Big)^s\zeta(s,\mathfrak{C}).$$

Finally, since $L=\mathbb{Q}(\sqrt{D})$ and $\mathfrak{m}$ is given by (4.16) with $\mathfrak{f}=(N)\subset\mathcal{O}_L$, then in Theorem C.2.2 of Appendix C, we have $r_1=2$, $r_2=0$, $s(\mathfrak{m})=2$, $R_\mathfrak{m}=\log(\varepsilon_\gamma)$ with $U_\mathfrak{m}=\langle\varepsilon_\gamma\rangle$, and $\mathcal{N}\mathfrak{m}=2^2N^2$, and therefore, we obtain

$$\zeta(s,\mathfrak{C}) = \frac{\log(\varepsilon_\gamma)}{N^2\sqrt{D}}\frac{1}{s-1} + O(1),$$

which in turn implies

$$\Big(\frac{1}{N}\Big)^s\zeta(s,\mathfrak{C}) = \frac{\log(\varepsilon_\gamma)}{N^3\sqrt{D}}\frac{1}{s-1} + O(1).$$

Thus, we have

$$\mathrm{res}_{s=1}\Big(\zeta_\gamma(s)\Big) = \frac{\log(\varepsilon_\gamma)}{N^3\sqrt{D}}.$$

This concludes the proof. □

Remark 4.2.12. It seems to be possible to extend the method used in the proof of Proposition 4.2.11 to the case $\Gamma_1(N)$. More precisely, we conjecture the following: Let N and l as in Proposition 4.2.11. Let $\gamma\in\mathrm{sp}_l(\Gamma_1(N))$ be a matrix of the form

$$\gamma = \begin{pmatrix} \frac{l-bN}{2} & -c \\ aN & \frac{l+bN}{2}\end{pmatrix}$$

with $a, b, c \in \mathbb{Z}$ such that $\gcd(aN, bN, c) = 1$ and $D = (bN)^2 - 4acN$ is not a square. Suppose that $\mathfrak{m}$ is the cycle of $L := \mathbb{Q}(\sqrt{D})$ given by (4.16) with $\mathfrak{f} = (N)$ and $\mathfrak{b} = \mathbb{Z} + \mathbb{Z}\theta$ with $\theta = -(bN + \sqrt{D})/2c$. Let $\mathfrak{C}_1$ denote the $\mathfrak{m}$-ideal class of $(u/N)\mathfrak{b}^{-1}\mathfrak{f}$, for some $1 \leq u \leq N$ with $\gcd(u, N) = 1$. Then the following identity holds

$$\zeta_\gamma(s) = \zeta(s, \mathfrak{C}_1).$$

Here, we have considered the set $M_{\Gamma_1(N)} = M_{\Gamma(N)}$ (see Notation 4.2.4). This would be a new method to proof [May14, Proposition 4.2.2, p. 130].

Remark 4.2.13. Similarly to the above remark, we conjecture the following conjecture for the congruence subgroup $\Gamma_0(N)$: Let N be an odd square-free positive integer and $l \in \mathbb{Z}$ such that $|l| > 2$. Fix a divisor $d|N$ of N. Let $\gamma \in \mathrm{sp}_l(\Gamma_0(N))$ be a matrix of the form

$$\gamma = \begin{pmatrix} \frac{l-b}{2} & -c \\ aN & \frac{l+b}{2} \end{pmatrix}$$

with $a, b, c \in \mathbb{Z}$ such that $\gcd(aN, b, c) = 1$ and $D = b^2 - 4acN$ is not a square. Suppose that $\mathfrak{m}$ is the cycle of $L := \mathbb{Q}(\sqrt{D})$ given by (4.16) with $\mathfrak{f} = (N)$ and $\mathfrak{b} = \mathbb{Z} + \mathbb{Z}\theta$ with $\theta = (b + \sqrt{D})/2ad$. Let $e \in \mathbb{Z}$ such that $(e)\mathfrak{b} \subset \mathcal{O}_L$ and let $\mathfrak{C}_0$ denote the $\mathfrak{m}$-ideal class of $(eN)\mathfrak{b}^{-1}$. Then the following identity holds

$$\zeta_\gamma(s) = \left(\frac{N}{d}\right)^s \zeta(s, \mathfrak{C}_0).$$

Here, we have considered the set $M_{\Gamma_0(N)} = \{(Nm, dn) \mid (m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}\}$. Again, this would be a new method to obtain [AU97, Proposition 3.2.3, p. 32] for the case $|l| > 2$.

4.3 The hyperbolic contribution. II

In this section we will give the proof of Proposition 4.1.15. To do this, we will first give a formula for the term $\int_{\mathcal{F}_\Gamma} \mathcal{H}(z) E_\infty^\Gamma(z, s)\mu_{\mathrm{hyp}}(z)$ in (4.12) (see Corollary 4.3.4). With this we will be able to find the expansion at $s = 1$ and as a result, we will obtain the hyperbolic contribution $\mathscr{R}_\infty^{\mathrm{hyp}}$.

Let $l \in \mathbb{Z}$ such that $|l| > 2$ and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of genus $g_\Gamma > 1$. Suppose that we have the following data associated to Γ: a finite set Ω_Γ and for each $\nu \in \Omega_\Gamma$ a subset $M_\Gamma(\nu) \subset \mathbb{Z}^2 \setminus \{(0, 0)\}$ which is Γ-invariant by right multiplication. Therefore, for $\gamma \in \mathrm{sp}_l(\Gamma)$ and $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, we have

$$\zeta_{\gamma,\nu}(s) := \sum_{(m,n) \in M_\Gamma^{f_\gamma}(\nu)/\mathrm{Stab}_\Gamma(f_\gamma)} f_\gamma(n, -m)^{-s}.$$

In particular, we fix once for all the following data for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$:

(i) If $\Gamma = \Gamma_0(N)$, then we take

$$\Omega_{\Gamma_0(N)} = \{d \in \mathbb{N} \mid d|N\};$$

$$M_{\Gamma_0(N)}(d) = \{(Nm, dn) \mid (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}\},\ d \in \Omega_{\Gamma_0(N)};$$

$$D_{\Gamma_0(N),d}(s) = \frac{\mu(d)}{2\zeta(2s)} \prod_{p|N} \left(1 - \frac{1}{p^{2s}}\right)^{-1},\ d \in \Omega_{\Gamma_0(N)}.$$

(ii) If $\Gamma = \Gamma_1(N)$, then we take

$$\Omega_{\Gamma_1(N)} = \{u \in \mathbb{N} \mid 1 \leq u \leq N,\ \gcd(u,N) = 1\};$$

$$M_{\Gamma_1(N)}(u) = \{(m,n) \in \mathbb{Z}^2 \mid (m,n) \equiv (0,u) \bmod N\},\ u \in \Omega_{\Gamma_1(N)};$$

$$D_{\Gamma_1(N),u}(s) = \sum_{\substack{d=1 \\ du\equiv 1 \bmod N}}^{\infty} \frac{\mu(d)}{d^{2s}},\ u \in \Omega_{\Gamma_1(N)}.$$

(iii) If $\Gamma = \Gamma(N)$, then we take

$$\Omega_{\Gamma(N)} = \{u \in \mathbb{N} \mid 1 \leq u \leq N,\ \gcd(u,N) = 1\};$$

$$M_{\Gamma(N)}(u) = \{(m,n) \in \mathbb{Z}^2 \mid (m,n) \equiv (0,u) \bmod N\},\ u \in \Omega_{\Gamma(N)};$$

$$D_{\Gamma(N),u}(s) = \frac{1}{N^s} \sum_{\substack{d=1 \\ du\equiv 1 \bmod N}}^{\infty} \frac{\mu(d)}{d^{2s}},\ u \in \Omega_{\Gamma(N)}.$$

Let $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. We define the functions

$$G_\Gamma(z,s) := \sum_{(m,n)\in M_\Gamma(\nu)} \frac{y^s}{|mz+n|^{2s}},$$

$$E(z,s) := \sum_{\nu\in\Omega_\Gamma} D_{\Gamma,\nu}(s) \cdot G_\Gamma(z,s),$$

where $D_{\Gamma,\nu}(s)$ is a function holomorphic at $s = 1$.

Remark 4.3.1. The choices made in (i)–(iii) implies $E(z,s) = E_\infty^\Gamma(z,s)$, provided that $\Gamma = \Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$ with $N \geq 3$ odd and square-free integer (see [AU97, Proposition 3.2.2, p. 29], [May14, p. 18], and Proposition 2.2.5). Furthermore, note that $D_{\Gamma_0(N),d}(s)$, $D_{\Gamma_1(N),u}(s)$, and $D_{\Gamma(N),u}(s)$ are indeed holomorphic at $s = 1$: For the first case is immediate and for the other two cases, observe that

$$\sum_{\substack{d=1 \\ du\equiv 1 \bmod N}}^{\infty} \frac{\mu(d)}{d^{2s}} = \frac{1}{\varphi(N)} \sum_{\chi \bmod N} \overline{\chi}(u^{-1}) \sum_{d=1}^{\infty} \frac{\chi(d)\mu(d)}{d^{2s}}$$

$$= \frac{1}{\varphi(N)} \sum_{\chi \bmod N} \overline{\chi}(u^{-1}) \frac{1}{L(2s,\chi)},$$

where in the first identity we used orthogonality relations of Dirichlet characters (see Appendix D.1) and in the second identity we used [Apo76, Example 3, p. 229]. Then the assertion follows because $L(2,\chi) \neq 0$ (see Remark 2.2.11).

Remark 4.3.2. By Proposition 4.2.11, we know that for the congruence subgroup $\Gamma(N)$, the function $\zeta_{\gamma,\nu}(s)$ has a Laurent expansion at $s=1$ of the form

$$\zeta_{\gamma,\nu}(s) = \frac{a_{-1}}{s-1} + a_0 + O(s-1),$$

where a_{-1} denotes the residue at $s=1$ and is equal to

$$a_{-1} = \frac{\log(\varepsilon_\gamma)}{N^3\sqrt{l^2-4}}.$$

We have similar Laurent expansions for the congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$, in which case the residues are equal to

$$\frac{\log(\varepsilon_\gamma)}{N\sqrt{l^2-4}} \quad \text{and} \quad \frac{\log(\varepsilon_\gamma)}{N^2\sqrt{l^2-4}},$$

respectively, (see [AU97] and [May14], respectively). Note that the definition of the zeta functions in the previous references varies with the one given here; thus we have adapted the residues accordingly.

Lemma 4.3.3. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_\Gamma > 1$. Suppose that $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < \inf\{A, 3/2\}$ and $l \in \mathbb{Z}$ with $|l| > 2$. Then the following identity holds*

$$\int\limits_{\mathcal{F}_\Gamma} \Bigg(\sum_{\gamma \in \mathrm{sp}_l(\Gamma)} \nu_k(\gamma; z) \Bigg) E(z,s)\mu_{\mathrm{hyp}}(z) = I_k(s;l) \sum_{\nu \in \Omega_\Gamma} D_{\Gamma,\nu}(s) \Bigg(\sum_{\gamma \in \mathrm{sp}_l(\Gamma)/\Gamma} \zeta_{\gamma,\nu}(s) \Bigg),$$

where $\nu_k(\gamma; z)$ is given by (4.10) *and $I_k(s;l)$ is given by*

$$I_k(s;l) := \int\limits_{\mathbb{H}} \left[\nu_k\left(\left(\begin{smallmatrix} l/2 & (l/2)^2-1 \\ 1 & l/2 \end{smallmatrix}\right); z \right) + \nu_k\left(\left(\begin{smallmatrix} -l/2 & (l/2)^2-1 \\ 1 & -l/2 \end{smallmatrix}\right); z \right) \right] \mathrm{Im}(z)^s \mu_{\mathrm{hyp}}(z).$$

Proof. Indeed, we have

$$\int\limits_{\mathcal{F}_\Gamma} \Bigg(\sum_{\gamma \in \mathrm{sp}_l(\Gamma)} \nu_k(\gamma; z) \Bigg) E(z,s)\mu_{\mathrm{hyp}}(z)$$

$$= \sum_{\nu \in \Omega_\Gamma} D_{\Gamma,\nu}(s) \times \int_{\mathcal{F}_\Gamma} \Bigg(\sum_{(\gamma,(m,n)) \in \mathrm{sp}_l(\Gamma) \times M_\Gamma(\nu)} \nu_k(\gamma; z) \frac{y^s}{|mz+n|^{2s}} \Bigg) \mu_{\mathrm{hyp}}(z).$$

Let $\overline{S}_\Gamma$ be the subset of $Q_l(\Gamma) \times M_\Gamma(\nu)$ defined by

$$\overline{S}_\Gamma = \{(f,(m,n)) \mid (m,n) \in M_\Gamma^{-f}(\nu) \; with \; f \in Q_l(\Gamma)\}.$$

Then we have the decomposition

$$Q_l(\Gamma) \times M_\Gamma(\nu) = S_\Gamma \sqcup \overline{S}_\Gamma,$$

where S_Γ is the set given in Lemma 4.2.5 (see, e.g., [May12, p. 95] for the special case $\Gamma = \Gamma_1(N)$, where Δ_l^{u+} is our set S_Γ). Now using the bijection $\mathrm{sp}_l(\Gamma) \simeq \rho(\mathrm{sp}_l(\Gamma)) = Q_l(\Gamma)$ (see fact (ii) before Lemma 4.2.3), we can write

$$\sum_{(\gamma,(m,n)) \in \mathrm{sp}_l(\Gamma) \times M_\Gamma(\nu)} \nu_k(\gamma; z) \frac{y^s}{|mz+n|^{2s}} \; = \; S_+(l,\nu,z) + S_-(l,\nu,z), \qquad (4.20)$$

where $S_+(l,\nu,z)$ and $S_-(l,\nu,z)$ are given by

$$S_+(l,\nu,z) = \sum_{(\gamma,(m,n)) \in S_\Gamma} \nu_k(\gamma; z) \frac{y^s}{|mz+n|^{2s}},$$

$$S_-(l,\nu,z) = \sum_{(\gamma,(m,n)) \in \overline{S}_\Gamma} \nu_k(\gamma; z) \frac{y^s}{|mz+n|^{2s}}.$$

Observe that by abuse of notation, we will write γ also for the binary quadratic form $f_\gamma = \rho(\gamma) \in Q_l(\Gamma)$. Now, note that

$$\begin{aligned} S_+(l,\nu,z) &= \sum_{(\gamma,(m,n)) \in S_\Gamma} \nu_k(\gamma; z) \frac{y^s}{|mz+n|^{2s}} \\ &= \sum_{(\gamma,(m,n)) \in S_\Gamma/\Gamma} \sum_{\alpha \in \Gamma} \nu_k(\alpha^{-1}\gamma\alpha; z) \frac{y^s}{|m_\alpha z + n_\alpha|^{2s}}, \end{aligned}$$

where $(m_\alpha, n_\alpha) = (m,n)\alpha$, i.e., it is the image of (m,n) under right multiplication by α. Similar considerations apply for $S_-(l,\nu,z)$. If we take now the integral of $S_+(l,\nu,z)$ over a fundamental domain $\mathcal{F}_\Gamma$ of Γ, then we have

$$\int_{\mathcal{F}_\Gamma} S_+(l,\nu,z) \mu_{\mathrm{hyp}}(z) = \int_{\mathcal{F}_\Gamma} \sum_{(\gamma,(m,n)) \in S_\Gamma/\Gamma} \sum_{\alpha \in \Gamma} \nu_k(\alpha^{-1}\gamma\alpha; z) \frac{y^s}{|m_\alpha z + n_\alpha|^{2s}}$$

Since $\nu_k(\alpha^{-1}\gamma\alpha; z) = \nu_k(\gamma; \alpha z)$ and

$$\frac{y}{|m_\alpha z + n_\alpha|^2} = \frac{\mathrm{Im}(\alpha z)}{|m(\alpha z) + n|^2},$$

by unfolding the integral, we obtain

$$\int\limits_{\mathcal{F}_\Gamma} S_+(l,\nu,z)\mu_{\mathrm{hyp}}(z) = \sum_{(\gamma,(m,n))\in S_\Gamma/\Gamma} \int\limits_{\mathbb{H}} \nu_k(\gamma;z)\frac{y^s}{|mz+n|^{2s}}\mu_{\mathrm{hyp}}(z)$$

$$= \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} \sum_{(m,n)\in M_\Gamma^{f_\gamma}(\nu)/\mathrm{Stab}_\Gamma(f_\gamma)} \int\limits_{\mathbb{H}} \nu_k(\gamma;z)\frac{y^s}{|mz+n|^{2s}}\mu_{\mathrm{hyp}}(z),$$

where in the second equality we used Lemma 4.2.6. Here, f_γ in the innermost sum is an actual binary quadratic form in $Q_l(\Gamma)$. Now, let M be the matrix given by

$$M = \frac{1}{f_\gamma(n,-m)^{1/2}}\begin{pmatrix} n & -(d-a)\frac{n}{2} - bm \\ -m & cn-(d-a)\frac{m}{2}\end{pmatrix},$$

where $f_\gamma = [c, d-a, -b]$. Making the change of variables

$$z \longmapsto Mz$$

in the integral of the right hand side, we obtain

$$\int\limits_{\mathcal{F}_\Gamma} S_+(l,\nu,z)\mu_{\mathrm{hyp}}(z) = I(l) \times \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} \sum_{(m,n)\in M_\Gamma^{f_\gamma}(\nu)/\mathrm{Stab}_\Gamma(f_\gamma)} f_\gamma(n,-m)^{-s},$$

where

$$I(l) = \int\limits_{\mathbb{H}} \nu_k\left(\left(\begin{smallmatrix} l/2 & l^2/4-1 \\ 1 & l/2\end{smallmatrix}\right); z\right) y^s \mu_{\mathrm{hyp}}(z).$$

The previous identity follows by using the identities

$$\frac{\mathrm{Im}(Mz)}{|m(Mz)+n|^2} = \frac{y}{f_\gamma(n,-m)}, \quad \text{and} \quad M^{-1}\gamma M = \begin{pmatrix} l/2 & l^2/4-1 \\ 1 & l/2\end{pmatrix}.$$

A similar identity holds for $S_-(l,\nu,z)$, namely, we have

$$\int\limits_{\mathcal{F}_\Gamma} S_-(l,\nu,z)\mu_{\mathrm{hyp}}(z) = I(-l) \times \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} \sum_{(m,n)\in M_\Gamma^{-f_\gamma}(\nu)/\mathrm{Stab}_\Gamma(-f_\gamma)} (-f_\gamma(n,-m))^{-s}$$

$$= I(-l) \times \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} \sum_{(m,n)\in M_\Gamma^{f_{\gamma^{-1}}}(\nu)/\mathrm{Stab}_\Gamma(f_{\gamma^{-1}})} (f_{\gamma^{-1}}(n,-m))^{-s}$$

$$= I(-l) \times \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} \sum_{(m,n)\in M_\Gamma^{f_\gamma}(\nu)/\mathrm{Stab}_\Gamma(f_\gamma)} (f_\gamma(n,-m))^{-s},$$

where in the second equality we used the identities $-f_\gamma = f_{-\gamma} = f_{\gamma^{-1}}$, which can be verified using fact (ii), p. 75.

The result follows by adding the integrals of $S_+(l,\nu,z)$ and $S_-(l,\nu,z)$ over the fundamental domain $\mathcal{F}_\Gamma$. □

Corollary 4.3.4. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_{\Gamma} > 1$. Suppose that $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < \inf\{A, 3/2\}$. Then the following identity holds*

$$\int_{\mathcal{F}_\Gamma} \mathcal{H}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) = \sum_{\substack{l\in\mathbb{Z}\\|l|>2}} \Big(I_2(s;l) - I_0(s;l)\Big) \sum_{\nu\in\Omega_\Gamma} \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} D_{\Gamma,\nu}(s)\zeta_{\gamma,\nu}(s).$$

Furthermore, the expansion at $s=1$ of the previous integral is given by

$$\frac{v_\Gamma^{-1}}{2\pi} \sum_{\substack{l\in\mathbb{Z}\\|l|>2}} \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_\gamma)}{\sqrt{l^2-4}} A_{|l|}(t) + O(s-1),$$

where ε_γ is a generator of $U_{\mathfrak{m}}$ (see Section 4.2, p. 82) and $A_{|l|}(t)$ is given by

$$A_{|l|}(t) = -\frac{\pi}{2\eta_l} h\Big(\frac{i}{2}\Big) + \frac{1}{4}\int_{-\infty}^{\infty} \frac{h(r)}{\frac{1}{4}+r^2} e^{-2ir\log(\eta_l)}\mathrm{d}r$$

with

$$\eta_l = \frac{|l| + \sqrt{l^2-4}}{2}.$$

Proof. First of all, we rewrite $\mathcal{K}_k^{\mathrm{hyp}}(z)$ in a more convenient way as follows

$$\mathcal{K}_k^{\mathrm{hyp}}(z) = \frac{1}{2} \sum_{\substack{\gamma\in\{\pm I\}\Gamma\\|\mathrm{tr}(\gamma)|>2}} \nu_k(\gamma;z) = \frac{1}{2} \sum_{\substack{l\in\mathbb{Z}\\|l|>2}} \sum_{\gamma\in\mathrm{sp}_l(\Gamma)} \nu_k(\gamma;z).$$

Then we have

$$\mathcal{R}_\infty^\Gamma[\mathcal{K}_k^{\mathrm{hyp}}](s) = \frac{1}{2} \sum_{\substack{l\in\mathbb{Z}\\|l|>2}} \int_{\mathcal{F}_\Gamma} \Bigg(\sum_{\gamma\in\mathrm{sp}_l(\Gamma)} \nu_k(\gamma;z) \Bigg) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z),$$

and by Lemma 4.3.3, we obtain

$$\mathcal{R}_\infty^\Gamma[\mathcal{K}_k^{\mathrm{hyp}}](s) = \frac{1}{2} \sum_{\substack{l\in\mathbb{Z}\\|l|>2}} I_k(s;l) \sum_{\nu\in\Omega_\Gamma} \Bigg(D_{\Gamma,\nu}(s) \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} \zeta_{\gamma,\nu}(s) \Bigg).$$

Consequently, we have

$$\begin{aligned}\int_{\mathcal{F}_\Gamma} \mathcal{H}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) &= \int_{\mathcal{F}_\Gamma} \Big(\mathcal{K}_2^{\mathrm{hyp}}(z) - \mathcal{K}_0^{\mathrm{hyp}}(z)\Big) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) \\ &= \frac{1}{2} \sum_{\substack{l\in\mathbb{Z}\\|l|>2}} \Big(I_2(s;l) - I_0(s;l)\Big) \sum_{\nu\in\Omega_\Gamma} \Bigg(D_{\Gamma,\nu}(s) \sum_{\gamma\in\mathrm{sp}_l(\Gamma)/\Gamma} \zeta_{\gamma,\nu}(s) \Bigg).\end{aligned}$$

Now, since we have expansions at $s = 1$ of the form

$$\begin{aligned}
\zeta_{\gamma,\nu}(s) &= \frac{a_{-1}}{s-1} + a_0 + O(s-1),\\
D_{\Gamma,\nu}(s) &= d_{\Gamma,\nu} + O(s-1),\\
I_2(s;l) - I_0(s;l) &= A_{|l|}(t)(s-1) + O((s-1)^2),
\end{aligned}$$

where a_{-1} refers to the residue at $s = 1$ of $\zeta_{\gamma,\nu}(s)$ (see Remark 4.3.2), $d_{\Gamma,\nu}$ is given by

$$d_{\Gamma,\nu} = \begin{cases} \dfrac{\mu(d)}{2N\zeta(2)} \displaystyle\prod_{p|N}\left(1-\frac{1}{p^2}\right)^{-1}, & \Gamma = \Gamma_0(N);\\ \displaystyle\sum_{\substack{1\le u\le N\\ \gcd(u,N)=1}} \sum_{\substack{d\ge 1\\ du\equiv 1 \bmod N}} \frac{\mu(d)}{d^2}, & \Gamma = \Gamma_1(N);\\ \dfrac{1}{N} \displaystyle\sum_{\substack{1\le u\le N\\ \gcd(u,N)=1}} \sum_{\substack{d\ge 1\\ du\equiv 1 \bmod N}} \frac{\mu(d)}{d^2}, & \Gamma = \Gamma(N), \end{cases}$$

and for the expansion of $I_2(s;l) - I_0(s;l)$, we refer the reader to Lemma B.2.2 of Appendix B. Therefore, we have

$$\int\limits_{\mathcal{F}_\Gamma} \mathcal{H}(z) E^\Gamma_\infty(z,s)\mu_{\mathrm{hyp}}(z) = \frac{1}{2}\sum_{\substack{l\in\mathbb{Z}\\ |l|>2}} \sum_{\gamma\in \mathrm{sp}_l(\Gamma)} A_{|l|}(t) a_{-1} V_\Gamma + O(s-1),$$

where

$$V_\Gamma := \sum_{\nu\in\Omega} d_{\Gamma,\nu}.$$

We claim that $V_\Gamma = v_\Gamma^{-1}/\pi$. Indeed, since we have

$$V_\Gamma = \begin{cases} \dfrac{1}{2N\zeta(2)} \left[\displaystyle\prod_{p|N}\left(1-\frac{1}{p^2}\right)^{-1}\right] \displaystyle\sum_{d|N}\frac{\mu(d)}{d}, & \Gamma = \Gamma_0(N);\\ \dfrac{1}{N^2} \displaystyle\sum_{\substack{1\le u\le N\\ \gcd(u,N)=1}} \sum_{\substack{d\ge 1\\ du\equiv 1 \bmod N}} \frac{\mu(d)}{d^2}, & \Gamma = \Gamma_1(N);\\ \dfrac{1}{N^3} \displaystyle\sum_{\substack{1\le u\le N\\ \gcd(u,N)=1}} \sum_{\substack{d\ge 1\\ du\equiv 1 \bmod N}} \frac{\mu(d)}{d^2}, & \Gamma = \Gamma(N), \end{cases}$$

the identities

$$\sum_{d|N}\frac{\mu(d)}{d} = \prod_{p|N}\left(1-\frac{1}{p}\right),$$

$$\sum_{\substack{1\leq u\leq N\\ \gcd(u,N)=1}} \sum_{\substack{d\geq 1\\ du\equiv 1 \bmod N}} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} \prod_{p|N} \left(1-\frac{1}{p^2}\right)^{-1}$$

(for the last identity, see [May14, (4.10), p. 131]), imply the assertion. Hence, we obtain

$$\int\limits_{\mathcal{F}_\Gamma} \mathcal{H}(z) E_\infty^\Gamma(z,s) \mu_{\mathrm{hyp}}(z) = \frac{v_\Gamma^{-1}}{2\pi} \sum_{\substack{l\in\mathbb{Z}\\ |l|>2}} \sum_{\gamma\in \mathrm{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_\gamma)}{\sqrt{l^2-4}} A_{|l|}(t) + O(s-1).$$

This concludes the proof. □

Proof of Proposition 4.1.15. By Corollary 4.3.4, we have

$$\mathscr{R}_\infty^{\mathrm{hyp}} = \frac{v_\Gamma^{-1}}{2\pi} \sum_{\substack{l\in\mathbb{Z}\\ |l|>2}} \sum_{\gamma\in \mathrm{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_\gamma)}{\sqrt{l^2-4}} A_{|l|}(t).$$

Now, by virtue of Lemma B.2.3 of Appendix B.2, we have

$$A_{|l|}(t) = -\frac{\pi}{2} \int\limits_0^t g_u(2\log(\eta_l)) \mathrm{d}u,$$

where

$$g_u(v) = \frac{1}{\sqrt{4\pi u}} e^{-\frac{1}{4}(\frac{v^2}{u}+u)}.$$

Consequently, we obtain

$$\begin{aligned}\sum_{\substack{l\in\mathbb{Z}\\ |l|>2}} \sum_{\gamma\in \mathrm{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_\gamma)}{\sqrt{l^2-4}} A_l(t) &= -\frac{\pi}{2} \int\limits_0^t \left(\sum_{\substack{l\in\mathbb{Z}\\ |l|>2}} \sum_{\gamma\in \mathrm{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_\gamma)}{\sqrt{l^2-4}} g_u(2\log(\eta_l))\right) \mathrm{d}u \\ &= -\frac{\pi}{2} \int\limits_0^t \Theta_\Gamma(u) \mathrm{d}u.\end{aligned}$$

This concludes the proof. □

4.4 The parabolic contribution

In this section we will give the proof of Proposition 4.1.16. To do this, we will first give a formula for the term $\int_{\mathcal{F}_\Gamma} \mathcal{P}(z) E_\infty^\Gamma(z,s) \mu_{\mathrm{hyp}}(z)$ in (4.12) (see Corollary 4.4.10) and with this, we will determine the expansion at $s=1$ and as a result, we will obtain the parabolic contribution $\mathscr{R}_\infty^{\mathrm{par}}$.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of genus $g_\Gamma > 1$ and suppose that $k = 0, 2$. Recall that

$$\mathcal{P}(z) = \mathcal{K}_2^{\mathrm{par}}(z) - \mathcal{K}_0^{\mathrm{par}}(z)$$

(see Section 4.1, p. 72) with

$$\mathcal{K}_k^{\mathrm{par}}(z) = \frac{1}{2}\Bigg(\sum_{\substack{\gamma\in\{\pm I\}\Gamma\\ |\mathrm{tr}(\gamma)|=2}} \nu_k(\gamma; z)\Bigg) - S_k(z),$$

where $S_k(z)$ is given by

$$S_k(z) := \frac{1}{4\pi}\sum_{q\in C_\Gamma}\int\limits_{-\infty}^{\infty} h(r)\Big|E_{q,k}^{\Gamma}\Big(z, \frac{1}{2}+ir\Big)\Big|^2 \mathrm{d}r + \frac{2-k}{2}v_{\Gamma}^{-1}.$$

Now, suppose that $z = x+iy \in \mathbb{H}$ and set

$$p_1(y;k) := \frac{1}{2}\int\limits_{-1/2}^{1/2}\Bigg(\sum_{\substack{\gamma\in\{\pm I\}\Gamma\smallsetminus\{\pm I\}\Gamma_\infty\\ |\mathrm{tr}(\gamma)|=2}} \nu_k(\gamma;\sigma_\infty z)\Bigg)\mathrm{d}x;$$

$$p_2(y;k) := \frac{1}{2}\Bigg(\sum_{\gamma\in\{\pm I\}\Gamma_\infty} \nu_k(\gamma;\sigma_\infty z)\Bigg) - \frac{y}{2\pi}\int\limits_{-\infty}^{\infty} h(r)\mathrm{d}r;$$

$$p_3(y;k) := -\frac{y}{2\pi}\int\limits_{-\infty}^{\infty} h(r)\varphi_{\infty\infty}^{\Gamma}\Big(\frac{1}{2}-ir\Big)\left[\frac{\frac{1}{2}+ir}{\frac{1}{2}-ir}\right]^{k/2} y^{2ir}\mathrm{d}r - \frac{2-k}{2}v_{\Gamma}^{-1};$$

$$p_4(y;k) := -\int\limits_{-1/2}^{1/2}\left[\frac{1}{4\pi}\int\limits_{-\infty}^{\infty} h(r)\sum_{q\in C_\Gamma}\Big|\widetilde{E}_{q,k}^{\Gamma}\Big(\sigma_\infty z, \frac{1}{2}+ir\Big)\Big|^2\mathrm{d}r\right]\mathrm{d}x;$$

$$p_4^*(y) := -\frac{1}{4\pi}\sum_{q\in C_\Gamma}\int\limits_{-\infty}^{\infty}\left(\frac{h(r)}{\frac{1}{4}+r^2}\mathcal{R}_\infty^{\Gamma}\Big[\Big|\widetilde{E}_{q,k}^{\Gamma}\Big(\sigma_\infty\cdot, \frac{1}{2}+ir\Big)\Big|^2\Big](s)\right)\mathrm{d}r; \tag{4.21}$$

where $\widetilde{E}_{q,k}^{\Gamma}(z,s)$ is given by

$$\widetilde{E}_{q,k}^{\Gamma}(z,s) := E_{q,k}^{\Gamma}(z,s) - \delta_{q\infty}y^s - \varphi_{q\infty,k}^{\Gamma}(s)y^{1-s}.$$

In what follows, we denote by $\mathcal{M}_j(s)$ ($j = 1,2,3,4$) the following Mellin transform (see Appendix A.1)

$$\mathcal{M}_j(s) := \int\limits_0^{\infty}\Big(p_j(y;2) - p_j(y;0)\Big)y^{s-2}\mathrm{d}y.$$

Lemma 4.4.1. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_\Gamma > 1$. Suppose that $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < A$. Then the following identity holds*

$$\int\limits_{\mathcal{F}_\Gamma}\mathcal{P}(z)E_\infty^{\Gamma}(z,s)\mu_{\mathrm{hyp}}(z) = \mathcal{M}_1(s)+\mathcal{M}_2(s)+\mathcal{M}_3(s)+\mathcal{M}_4(s).$$

Proof. For the proof let us simply write σ_∞ to denote the scaling matrix σ_∞^Γ of the cusp $\infty \in C_\Gamma$.

First of all note that the Rankin–Selberg method (see [Zag81a, p. 314]) gives the identity

$$\int\limits_{\mathcal{F}_\Gamma} \mathcal{K}_k^{\mathrm{par}}(z) E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) = \int\limits_0^\infty p(y;k)y^{s-2}\mathrm{d}y,$$

where $p(y;k)$ denotes the 0-th Fourier coefficient of $\mathcal{K}_k^{\mathrm{par}}(\sigma_\infty z)$, i.e., we have

$$p(y;k) = \int\limits_{-1/2}^{1/2} \mathcal{K}_k^{\mathrm{par}}(\sigma_\infty z)\mathrm{d}x = I(y) - J(y),$$

where

$$I(y) := \frac{1}{2}\int\limits_{-1/2}^{1/2} \Big(\sum_{\substack{\gamma \in \{\pm I\}\Gamma \\ |\mathrm{tr}(\gamma)|=2}} \nu_k(\gamma;\sigma_\infty z)\Big)\mathrm{d}x,$$

$$J(y) := \int\limits_{-1/2}^{1/2} S_k(\sigma_\infty z)\mathrm{d}x.$$

Secondly, note that

$$\begin{aligned} I(y) &= \frac{1}{2}\int\limits_{-1/2}^{1/2} \Big(\sum_{\substack{\gamma \in \{\pm I\}\Gamma \smallsetminus \{\pm I\}\Gamma_\infty \\ |\mathrm{tr}(\gamma)|=2}} \nu_k(\gamma;\sigma_\infty z)\Big)\mathrm{d}x + \frac{1}{2}\int\limits_{-1/2}^{1/2} \Big(\sum_{\gamma \in \{\pm I\}\Gamma_\infty} \nu_k(\gamma;\sigma_\infty z)\Big)\mathrm{d}x \\ &= p_1(y;k) + \frac{1}{2}\sum_{\gamma \in \{\pm I\}\Gamma_\infty} \nu_k(\gamma;\sigma_\infty z), \end{aligned}$$

where in the second equality we used the fact that $\nu_k(\gamma;\sigma_\infty z)$ does not depend on the variable x provided that $\gamma \in \{\pm I\}\Gamma_\infty$ (see proof of Lemma 4.4.7).

Thirdly, we have

$$\begin{aligned} J(y) &= H(y) + \frac{1}{4\pi}\int\limits_{-1/2}^{1/2}\int\limits_{-\infty}^{\infty} \Big(h(r)\sum_{q \in C_\Gamma} \Big| \widetilde{E}_{q,k}^\Gamma\Big(\sigma_\infty z, \frac{1}{2}+ir\Big)\Big|^2 \Big)\mathrm{d}r\,\mathrm{d}x + \frac{2-k}{2}v_\Gamma^{-1} \\ &= H(y) - p_4(y;k) + \frac{2-k}{2}v_\Gamma^{-1}, \end{aligned}$$

where

$$H(y) := \frac{1}{4\pi}\int\limits_{-1/2}^{1/2}\int\limits_{-\infty}^{\infty} \Big(h(r)\sum_{q \in C_\Gamma} \Big| a_0\Big(y, \frac{1}{2}+ir, q\infty; k\Big)\Big|^2 \Big)\mathrm{d}r\,\mathrm{d}x$$

and $a_0(y,s,q\infty;k)$ denotes the 0-th Fourier coefficient of $E^{\Gamma}_{q,k}(z,s)$ at ∞, i.e., we have

$$a_0(y,s,q\infty;k) = \delta_{q\infty} y^s + \varphi^{\Gamma}_{q\infty,k}(s) y^{1-s}.$$

Fourthly, using part (b) of Lemma 1.4.12, we obtain

$$\begin{aligned} H(y) &= \frac{y}{2\pi}\int_{-\infty}^{\infty} h(r)\mathrm{d}r + \frac{y}{2\pi}\int_{-\infty}^{\infty} h(r)\varphi^{\Gamma}_{\infty\infty}\Big(\frac{1}{2}-ir\Big)\left[\frac{\frac{1}{2}+ir}{\frac{1}{2}-ir}\right]^{k/2} y^{2ir}\mathrm{d}r \\ &= \frac{1}{2}\sum_{\gamma\in\{\pm I\}\Gamma_\infty} \nu_k(\gamma;\sigma_\infty z) - p_2(y;k) - p_3(y;k) - \frac{2-k}{2}v_{\Gamma}^{-1}. \end{aligned}$$

Finally, we have

$$\begin{aligned} p(y;k) &= I(y) - J(y) \\ &= p_1(y;k) + \frac{1}{2}\sum_{\gamma\in\{\pm I\}\Gamma_\infty} \nu_k(\gamma;\sigma_\infty z) - H(y) + p_4(y;k) - \frac{2-k}{2}v_{\Gamma}^{-1} \\ &= p_1(y;k) + p_2(y;k) + p_3(y;k) + p_4(y;k). \end{aligned}$$

From this, the result follows. $\square$

In the following lemmas we will provide formulas for $\mathcal{M}_j(s)$, where $j=1,2,3,4$. In addition, we also compute the Laurent expansion of $\mathcal{M}_j(s)$ at $s=1$.

Notation 4.4.2. Let $m,c,d\in\mathbb{Z}$ be integers. In the sequel, we denote by $\gamma_{\genfrac{}{}{0pt}{}{+}{(-)}}(m;c,d)$ the matrix given by

$$\gamma_{\genfrac{}{}{0pt}{}{+}{(-)}}(m;c,d) := \begin{pmatrix} \genfrac{}{}{0pt}{}{+}{(-)}1+mcd & md^2 \\ -mc^2 & \genfrac{}{}{0pt}{}{+}{(-)}1-mcd \end{pmatrix}.$$

Also, we write $P := \{\pm I\}B$, where B is given by Notation 1.2.11.

Lemma 4.4.3. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N\geq 3$ is an odd square-free integer such that $g_{\Gamma}>1$. Suppose that $s\in\mathbb{C}$ with $1<\mathrm{Re}(s)<A$. Then the following identity holds*

$$\begin{aligned} &\frac{1}{2}\int_{-1/2}^{1/2}\int_{0}^{\infty}\Bigg(\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in P\backslash \mathrm{SL}_2(\mathbb{Z})\\ \gamma_{\genfrac{}{}{0pt}{}{+}{(-)}}(m;c,d)\in\{\pm I\}\Gamma}} \nu_k\Big(\gamma_{\genfrac{}{}{0pt}{}{+}{(-)}}(m;c,d);\sigma_\infty z\Big)\Bigg) y^{s-2}\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\sideset{}{'}\sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)} \frac{(w^{\Gamma}_{\infty})^{-s}}{(mc^2)^s}\int_{\mathbb{H}}\nu_k\left(\begin{pmatrix} \genfrac{}{}{0pt}{}{+}{(-)}\mathrm{sign}(m) & 0 \\ 1 & \genfrac{}{}{0pt}{}{+}{(-)}\mathrm{sign}(m)\end{pmatrix};z\right)\mathrm{Im}(z)^s\mu_{\mathrm{hyp}}(z), \end{aligned}$$

where $\nu_k(\gamma; z)$ is given by (4.10), the sum $\sum'$ runs over all representatives $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ of the double coset $B\backslash\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}\Gamma_\infty$ such that $\gamma_{\overset{+}{(-)}}(m; c, d) \in \{\pm I\}\Gamma$.

Proof. For the proof, first of all note that the integrand

$$\frac{1}{2}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\ \sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in P\backslash\mathrm{SL}_2(\mathbb{Z})\\ \gamma_{\overset{+}{(-)}}(m;c,d)\in\{\pm I\}\Gamma}} \nu_k\Big(\gamma_{\overset{+}{(-)}}(m; c, d); \sigma_\infty z\Big)$$

on the left hand side of the previous identity can be rewritten as follows

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\ \sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in B\backslash\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}\Gamma_\infty\\ \gamma_{\overset{+}{(-)}}(m;c,d)\in\{\pm I\}\Gamma}}\ \sum_{n\in\mathbb{Z}} \nu_k\Big(\gamma_{\overset{+}{(-)}}(m; c, d); w_\infty^\Gamma(x+n) + iw_\infty^\Gamma y\Big),$$

where w_∞^Γ denotes the width of the cusp ∞.

Secondly, we apply the following sequence of change of variables: $x \longmapsto x - n$, $x \longmapsto (x/w_\infty^\Gamma)$, and $y \longmapsto (y/w_\infty^\Gamma)$. As a result, we obtain

$$\frac{1}{2}\int\limits_{-1/2}^{1/2}\int\limits_{0}^{\infty}\Bigg(\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\ \sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in P\backslash\mathrm{SL}_2(\mathbb{Z})\\ \gamma_{\overset{+}{(-)}}(m;c,d)\in\{\pm I\}\Gamma}} \nu_k\Big(\gamma_{\overset{+}{(-)}}(m; c, d); \sigma_\infty z\Big)\Bigg) y^{s-2}\mathrm{d}x\,\mathrm{d}y$$

$$= \Big(w_\infty^\Gamma\Big)^{-s}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\ \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)}{}' \int\limits_{\mathbb{H}} \nu_k\Big(\gamma_{\overset{+}{(-)}}(m; c, d); z\Big)\mathrm{Im}(z)^s\mu_{\mathrm{hyp}}(z).$$

Finally, by an immediate extension of [AU97, Lemme 3.2.12, p. 37], we have

$$\int\limits_{\mathbb{H}} \nu_k\Big(\gamma_{\overset{+}{(-)}}(m; c, d); z\Big)\mathrm{Im}(z)^s\mu_{\mathrm{hyp}}(z)$$

$$= \frac{1}{|mc^2|^s}\int\limits_{\mathbb{H}} \nu_k\Bigg(\begin{pmatrix} \overset{+}{(-)}\mathrm{sign}(m) & 0 \\ 1 & \overset{+}{(-)}\mathrm{sign}(m) \end{pmatrix}; z\Bigg)\mathrm{Im}(z)^s\mu_{\mathrm{hyp}}(z).$$

This concludes the proof. $\square$

Lemma 4.4.4. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_\Gamma > 1$. Suppose that $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < A$. Then the following identity holds*

$$\mathcal{M}_1(s) = \Big(w_\infty^\Gamma\Big)^{-s}[I_2(s;2) - I_0(s;2)]\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\ \sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in B\backslash\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}\Gamma_\infty\\ \gamma_{\pm}(m;c,d)\in\{\pm I\}\Gamma}} \frac{1}{|mc^2|^s},$$

where $\nu_k(\gamma; z)$ is given by (4.10) *and $I_k(s;2)$ is given by*

$$I_k(s;2) := \int\limits_{\mathbb{H}} \Big[\nu_k\Big(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right); z\Big) + \nu_k\Big(\left(\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right); z\Big)\Big] \mathrm{Im}(z)^s \mu_{\mathrm{hyp}}(z),$$

and the innemost sum runs over all representatives $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ of classes in the double quotient $B\backslash \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}\Gamma_\infty$ such that either $\gamma_+(m;c,d)$ or $\gamma_-(m;c,d)$ belongs to $\{\pm I\}\Gamma$.

Proof. For the proof, note that a matrix $\gamma \in \{\pm I\}\Gamma$ with $\mathrm{tr}(\gamma) = \pm 2$ can be written as follows

$$\gamma = \sigma_\gamma^{-1} \begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} \sigma_\gamma,$$

where $m \in \mathbb{Z}$ with $m \neq 0$ and $\sigma_\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is uniquely determined by γ up to left multiplication by an element in P (see [AU97, p. 37]). Therefore, if $\sigma_\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $\gamma = \gamma_{\underset{(-)}{+}}(m; c, d)$ and with this we have

$$\sum_{\substack{\gamma \in \{\pm I\}\Gamma \setminus \{\pm I\}\Gamma_\infty \\ |\mathrm{tr}(\gamma)| = 2}} \nu_k(\gamma; \sigma_\infty z) = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in P\backslash \mathrm{SL}_2(\mathbb{Z}) \\ \gamma_+(m;c,d) \in \{\pm I\}\Gamma}} \nu_k\Big(\gamma_\pm(m;c,d); \sigma_\infty z\Big)$$
$$+ \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in P\backslash \mathrm{SL}_2(\mathbb{Z}) \\ \gamma_-(m;c,d) \in \{\pm I\}\Gamma}} \nu_k\Big(\gamma_\pm(m;c,d); \sigma_\infty z\Big).$$

Now, we apply Lemma 4.4.3 and after regrouping terms, the result follows. This concludes the proof. □

Corollary 4.4.5. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_\Gamma > 1$. Suppose that $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < A$. Then the following assertions hold:*

(a) If $\Gamma = \Gamma_0(N)$, then $\mathcal{M}_1(s)$ is equal to

$$\zeta(s)\frac{\zeta(2s-1)}{\zeta(2s)}[I_2(s;2) - I_0(s;2)]\frac{1}{N^s}\Bigg(\frac{1}{\sigma_s(N)}\sum_{d|N}\varphi(N/d)\sigma_s(d)\Bigg).$$

The Laurent expansion of $\mathcal{M}_1(s)$ at $s = 1$ is given by

$$\frac{A_2(t)d(N)v_{\Gamma_0(N)}^{-1}}{s-1} + \frac{d(N)}{v_{\Gamma_0(N)}}\Bigg(C_1(t) + A_2(t)\Bigg[2\mathscr{C} + \gamma_{\mathrm{EM}}$$
$$+ \frac{1}{d(N)}\Big(\frac{\varphi(N)}{N} + 2^{\omega(N)-1}\log(N)\Big) + \sum_{p|N}\frac{1+2p}{1+p}\log(p)\Bigg]\Bigg)$$
$$+ O(s-1).$$

(b) If $\Gamma = \Gamma_1(N)$, then $\mathcal{M}_1(s)$ is equal to

$$\zeta(s)\frac{\zeta(2s-1)}{\zeta(2s)}[I_2(s;2) - I_0(s;2)]\frac{1}{N^s}\left(\frac{2}{\sigma_s(N)}\sum_{d|N}\sigma_s(d)\right).$$

The Laurent expansion of $\mathcal{M}_1(s)$ at $s=1$ is given by

$$\frac{A_2(t)\varphi(N)\prod_{p|N}(p+2)}{Nv_{\Gamma_0(N)}(s-1)} + \frac{\varphi(N)}{Nv_{\Gamma_0(N)}}\prod_{p|N}(p+2)\Bigg(C_1(t) + A_2(t)\Bigg[2\mathscr{C} + \gamma_{\mathrm{EM}}$$

$$+ \left(\prod_{p|N}\frac{1}{p+2}\right)\left(1 + \sum_{j=1}^{\omega(N)} 2^{\omega(N)-j}\sum_{\substack{d|N\\ \omega(d)=j}} d\log(d)\right) - \sum_{p|N}\frac{2p+1}{p+1}\log(p)\Bigg]\Bigg)$$

$$+ O(s-1).$$

(c) If $\Gamma = \Gamma(N)$, then $\mathcal{M}_1(s)$ is equal to

$$\zeta(s)\frac{\zeta(2s-1)}{\zeta(2s)}[I_2(s;2) - I_0(s;2)]\frac{1}{N^s}\left(2N^{1-s}\right).$$

The Laurent expansion of $\mathcal{M}_1(s)$ at $s=1$ is given by

$$\frac{12A_2(t)/\pi N}{s-1} + \frac{12}{\pi N}\left(C_1(t) + A_2(t)\left[2\mathscr{C} + \gamma_{\mathrm{EM}} - 2\log(N)\right]\right) + O(s-1).$$

Here, $A_2(t)$ is given by (B.2) *of Appendix B.2 and $C_1(t)$ depends only on the fixed real positive t, and it has finite limit $C_1 < \infty$ as $t \to \infty$.*

Proof. For the proof, we start with the case $\Gamma = \Gamma(N)$. Observe that the condition

$$\gamma_{\pm}(m;c,d) \in \{\pm I\}\Gamma(N)$$

is fulfilled if and only if $m \equiv 0 \bmod N$. Indeed, if $m \equiv 0 \bmod N$, then it is clear. Now suppose that

$$\begin{pmatrix} \pm 1 - mcd & md^2 \\ -mc^2 & \pm 1 + mcd \end{pmatrix} \in \{\pm I\}\Gamma(N),$$

where $m = ek$ for some $k \in \mathbb{Z}$, with $e|N$ and $\gcd(k, N/e) = 1$. Then it can be verified that $c \equiv d \equiv 0 \bmod (N/e)$. This will give a contradiction to the fact that $\gcd(c,d) = 1$, unless $e = N$.

Since we have

$$\begin{aligned}\#B\backslash \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}B(N) &= \#\{1 \le d \le cN \mid \gcd(c,d) = 1\}\\ &= N\varphi(c),\end{aligned}$$

we obtain from Lemma 4.4.4 the following

$$\begin{aligned}\mathcal{M}_1(s) &= N^{-s}[I_2(s;2) - I_0(s;2)] \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in B\backslash \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}\Gamma_\infty \\ \gamma_\pm(kN;c,d)\in\{\pm I\}\Gamma}} \frac{1}{|Nkc^2|^s} \\ &= 2\zeta(s)\frac{\zeta(2s-1)}{\zeta(2s)}[I_2(s;2) - I_0(s;2)]N^{1-2s}.\end{aligned}$$

For the Laurent expansion at $s = 1$, we use Lemma B.2.1 and (2.2), as well as the following

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} + \gamma_{\mathrm{EM}} + O(s-1), \\ N^{1-2s} &= \frac{1}{N} - \frac{2\log(N)}{N}(s-1) + O\big((s-1)^2\big).\end{aligned}$$

For the case $\Gamma = \Gamma_0(N)$, observe that the condition

$$\begin{pmatrix} \pm 1 - mcd & md^2 \\ -mc^2 & \pm 1 + mcd \end{pmatrix} \in \{\pm I\}\Gamma(N)$$

implies that $c \equiv 0 \bmod (N/e)$, where $e = \gcd(md^2, N)$. Thus, we recover [AU97, Lemme 3.2.11]. As for the Laurent expansion at $s = 1$, the following identity will be useful

$$\sum_{d|N}\left[\varphi(N/d)\sigma(d)\left(\sum_{p|d}\frac{p}{p+1}\log(p)\right)\right] = \varphi(N) + 2^{\omega(N)-1}N\log(N).$$

This holds only for N square-free and it can be proved using induction.

Finally, the case $\Gamma = \Gamma_1(N)$ is done in a similar way, and for the Laurent expansion at $s = 1$, we emphasize the following identity

$$\sum_{d|N}\left[\sigma(N)\left(\sum_{p|N}\frac{p}{p+1}\log(p)\right)\right] = 1 + \sum_{j=1}^{\omega(N)} 2^{\omega(N)-j} \sum_{\substack{d|N \\ \omega(d)=j}} d\log(d),$$

which holds for N square-free. This concludes the proof. $\square$

Notation 4.4.6. We set $\Psi_k^{\pm}(y) := \Psi_k(y) + \Psi_k(-y)$, where $\Psi_k(y)$ is the function given by

$$\Psi_k(y) := \int_{-\infty}^{\infty} \phi_k(u^2)\left(\frac{2-iu}{2+iu}\right)^{k/2} e^{2i\pi uy}\mathrm{d}u.$$

Here, ϕ_k stands for the inverse Selberg/Harish–Chandra transform of weight k of the function $h(r)$ given by (1.17).

Lemma 4.4.7. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index of genus $g_\Gamma > 1$. Suppose that $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < A$. Then the following identity holds*

$$\mathcal{M}_2(s) = \zeta(s)\Bigg[\int\limits_0^\infty \Psi_2^{\pm}(y)y^{s-1}\mathrm{d}y - \int\limits_0^\infty \Psi_0^{\pm}(y)y^{s-1}\mathrm{d}y\Bigg].$$

Furthermore, the Laurent expansion of $\mathcal{M}_2(s)$ at $s = 1$ is given by

$$\frac{C_2(t) + (4\pi)^{-1}}{s-1} + \left(\frac{1-\log(4\pi)}{4\pi} + \gamma_{\mathrm{EM}}C_2(t) + C_3(t)\right) + O(s-1),$$

where t is the fixed positive real in the definition of the function $h(r)$. Furthermore, $C_2(t)$ and $C_3(t)$ depend only on t and both tend to 0 as $t \to \infty$.

Proof. For the proof, note that

$$\frac{1}{2}\sum_{\gamma \in \{\pm I\}\Gamma_\infty} \nu_k(\gamma; \sigma_\infty z) = \frac{1}{2}\sum_{\gamma \in P} \nu_k(\gamma; z)$$

by virtue of Remark 4.1.10.

On the one hand, if $\gamma = \pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1\end{smallmatrix}\right) \in P$ then $j_\gamma(z;k) = 1$; on the other hand, if we write $z = x + iy$, then the following holds

$$\begin{aligned}\nu_k(\gamma; z) &= \left(\frac{z+n-\overline{z}}{z-\overline{z}-n}\right)^{k/2}\phi_k(u(z, z+n)) \\ &= \left(\frac{2-i\frac{n}{y}}{2+i\frac{n}{y}}\right)^{k/2}\phi_k\left(\frac{n^2}{y^2}\right).\end{aligned}$$

Consequently, we obtain

$$\frac{1}{2}\sum_{\gamma \in \{\pm I\}\Gamma_\infty} \nu_k(\gamma; \sigma_\infty z) = \sum_{n \in \mathbb{Z}} \left(\frac{2-i\frac{n}{y}}{2+i\frac{n}{y}}\right)^{k/2}\phi_k\left(\frac{n^2}{y^2}\right). \tag{4.22}$$

By the Poisson summation formula, the right hand side of the previous identity equals

$$\sum_{n\in\mathbb{Z}}\int\limits_{-\infty}^{\infty} \phi_k\left(\frac{v^2}{y^2}\right)\left(\frac{2-i\frac{v}{y}}{2+i\frac{v}{y}}\right)^{k/2} e^{2\pi inv}\mathrm{d}v.$$

Making the change of variables $u = -v/y$, we obtain

$$y\sum_{n\in\mathbb{Z}}\int\limits_{-\infty}^{\infty} \phi_k(u^2)\left(\frac{2+iu}{2-iu}\right)^{k/2} e^{2\pi inuy}\mathrm{d}u.$$

By convenience, we rewrite the previous expression as follows

$$y\int_{-\infty}^{\infty}\phi_k(u^2)\Big(\frac{2+iu}{2-iu}\Big)^{k/2}\mathrm{d}v+y\sum_{n\neq 0}\int_{-\infty}^{\infty}\phi_k(u^2)\Big(\frac{2+iu}{2-iu}\Big)^{k/2}e^{2\pi inuy}\mathrm{d}u.$$

Since we have

$$\int_{-\infty}^{\infty}\phi_k(u^2)\Big(\frac{2+iu}{2-iu}\Big)^{k/2}\mathrm{d}v=\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\mathrm{d}r$$

(see [AU97, (15) and (16), p. 24]), from (4.22) we obtain

$$\begin{aligned}y^{-1}p_2(y;k)&=\frac{1}{2y}\sum_{\gamma\in\{\pm I\}\Gamma_\infty}\nu_k(\gamma;\sigma_\infty z)-\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\mathrm{d}r\\&=\sum_{n\geq 1}\Big(\Psi_k(ny)+\Psi_k(-ny)\Big).\end{aligned}$$

Finally, we take the Mellin transform of $y^{-1}\Big(p_2(y;2)-p_0(y;0)\Big)$ and make the change of variables $y\mapsto ny$. Thus, we obtain the first assertion of the lemma.

For the Laurent expansion, we use lemmas B.3.1 and B.3.2 of Appendix B.3, and then proceed as in [May12, Proposition 5.3.2., p. 114]. This concludes the proof. □

Lemma 4.4.8. *Let $\Gamma\subset\mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index of genus $g_\Gamma>1$. Suppose that $s\in\mathbb{C}$ with $1<\mathrm{Re}(s)<A$. Then the following identity holds*

$$\mathcal{M}_3(s)=\Big(\frac{s}{1+s}\Big)h\Big(\frac{is}{2}\Big)\varphi^\Gamma_{\infty\infty}\Big(\frac{1+s}{2}\Big).$$

Furthermore, the Laurent expansion of $\mathcal{M}_3(s)$ at $s=1$ is given by

$$\frac{v_\Gamma^{-1}}{s-1}+\left(\frac{\mathscr{C}^\Gamma_{\infty\infty}}{2}+\frac{v_\Gamma^{-1}}{2}(t+1)\right)+O(s-1).$$

Proof. The proof of the first identity follows from an immediate extension of [AU97, Lemme 3.2.17, p. 44] and [May14, Lemma 5.1.1., p. 137] to subgroups of the modular group of finite index. For the Laurent expansion at $s=1$, we just multiply the next Laurent expansions at $s=1$

$$\begin{aligned}&\frac{1}{2}h\Big(\frac{is}{2}\Big)=\frac{1}{2}+\frac{t}{4}(s-1)+O\Big((s-1)^2\Big);\\&\varphi^\Gamma_{\infty\infty}\Big(\frac{1+s}{2}\Big)=\frac{2v_\Gamma^{-1}}{s-1}+\mathscr{C}^\Gamma_{\infty\infty}+O(s-1);\\&\frac{2s}{s+1}=1+\frac{1}{2}(s-1)+O\Big((s-1)^2\Big).\end{aligned}$$

This concludes the proof. $\square$

Lemma 4.4.9. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index of genus $g_\Gamma > 1$. Suppose that $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < A$. Then the following identity holds*

$$\mathcal{M}_4(s) = s\Big(\frac{s-1}{2}\Big)\int_0^\infty p_4^*(y)y^{s-2}\mathrm{d}y,$$

where $p_4^*(y)$ is given by (4.21). Furthermore, the Laurent expansion of $\mathcal{M}_4(s)$ at $s = 1$ is given by

$$C_4(t) + O(s-1),$$

where $C_4(t)$ depends only on fixed positive real t, and tends to 0 as $t \to \infty$.

Proof. The lemma follows from an immediate extension of [May14, Proposition 5.2.3, p. 141] to subgroups of the modular group of finite index. $\square$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, consider the function

$$\mathscr{G}_\Gamma(s) := \begin{cases} \dfrac{1}{\sigma_s(N)}\displaystyle\sum_{d|N}\varphi(N/d)\sigma_s(d), & \textit{if } \Gamma = \Gamma_0(N); \\ \dfrac{2}{\sigma_s(N)}\displaystyle\sum_{d|N}\sigma_s(d), & \textit{if } \Gamma = \Gamma_1(N); \\ 2N^{1-s}, & \textit{if } \Gamma = \Gamma(N). \end{cases} \tag{4.23}$$

Corollary 4.4.10. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, where $N \geq 3$ is an odd square-free integer such that $g_\Gamma > 1$. Suppose that $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < A$. Then the following identity holds*

$$\begin{aligned}&\int_{\mathcal{F}_\Gamma} \mathcal{P}(z)E_\infty^\Gamma(z,s)\mu_{\mathrm{hyp}}(z) \\ &= [I_2(s;2) - I_0(s;2)]\frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)}\frac{\mathscr{G}_\Gamma(s)}{N^s} + \frac{s}{s+1}h\Big(\frac{is}{2}\Big)\varphi_{\infty\infty}^\Gamma\Big(\frac{1+s}{2}\Big) \\ &+ \zeta(s)\Big[\int_0^\infty \Psi_2^\pm(y)y^{s-1}\mathrm{d}y - \int_0^\infty \Psi_0^\pm(y)y^{s-1}\mathrm{d}y\Big] + s\Big(\frac{s-1}{2}\Big)\int_0^\infty p_4^*(y)y^{s-2}\mathrm{d}y,\end{aligned}$$

where $\mathscr{G}_\Gamma(s)$ is given by (4.23) and the functions $I_k(s;2)$, $\Psi_k^\pm(y)$, and $p_4^*(y)$ are given by Lemma 4.4.4, Lemma 4.4.7 and (4.21), respectively.

Proof of Proposition 4.1.16. We prove the case $\Gamma = \Gamma(N)$. For the congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ the argument is similar.

First of all, we gather the following Laurent expansions of each $\mathcal{M}_j(s)$ at $s = 1$

proved in the previous lemmas

$$\mathcal{M}_1(s) = \frac{12A_2(t)}{\pi N(s-1)} + \frac{12}{\pi N}\Big(C_1(t) + A_2(t)\Big[2\mathscr{C} + \gamma_{\mathrm{EM}} - 2\log(N)\Big]\Big) + O(s-1),$$

$$\mathcal{M}_2(s) = \frac{C_2(t) + (4\pi)^{-1}}{s-1} + \left(\frac{1-\log(4\pi)}{4\pi} + \gamma_{\mathrm{EM}}C_2(t) + C_3(t)\right) + O(s-1),$$

$$\mathcal{M}_3(s) = \frac{v_{\Gamma(N)}^{-1}}{s-1} + \left(\frac{\mathscr{C}_{\infty\infty}^{\Gamma(N)}}{2} + \frac{v_{\Gamma}^{-1}}{2}(t+1)\right) + O(s-1),$$

$$\mathcal{M}_4(s) = C_4(t) + O(s-1).$$

Now, by adding these expansions, we obtain the Laurent expansion of the integral $\int_{\mathcal{F}_{\Gamma(N)}} \mathcal{P}(z)E_{\infty}^{\Gamma(N)}\mu_{\mathrm{hyp}}(z)$ at $s=1$. Since $\mathscr{R}_{\infty}^{\mathrm{par}}$ denotes the constant term in this expansion, we have

$$\begin{aligned}\mathscr{R}_{\infty}^{\mathrm{par}} = {} & \frac{12}{\pi N}\Big(C_1(t) + A_2(t)\Big[2\mathscr{C} + \gamma_{\mathrm{EM}} - 2\log(N)\Big]\Big) + \frac{1-\log(4\pi)}{4\pi} \\ & + \gamma_{\mathrm{EM}}C_2(t) + C_3(t) + \frac{\mathscr{C}_{\infty\infty}^{\Gamma(N)}}{2} + \frac{v_{\Gamma(N)}^{-1}}{2}(t+1) + C_4(t).\end{aligned}$$

The result follows after noting that

$$\frac{N\varphi(N)}{v_{\Gamma(N)}}\prod_{p|N}\left(1+\frac{1}{p}\right) = \frac{6}{\pi N}$$

and reordering terms. This concludes the proof. $\square$

Chapter 5

Geometric contribution and asymptotic expansion of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$

In the first half of this chapter, we investigate $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ for certain class of congruence subgroups Γ. Also, we describe a method for determining the geometric contribution using a morphism $\pi : \mathcal{X}_{\Gamma(N)} \longrightarrow \mathcal{X}_\Gamma$. In the second half, we analyze the asymptotic behaviour of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}/[K:\mathbb{Q}]$ as the level N tends to infinity, for the congruence subgroups $\Gamma = \Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$.

In Section 5.1, we introduce the Tate curve and the closed subscheme of cusps of $\mathcal{X}_\Gamma/\mathcal{O}_K$. As a result, we obtain a parametrization of the irreducible components of the fibers $(\mathcal{X}_{\Gamma(N)})_\mathfrak{p}$ with $\mathfrak{p}|N$ and with this a justification of the method mentioned in the previous paragraph.

In Section 5.2, we proceed to the explicit calculations of the geometrical parts and the analysis the asymptotic behaviour of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ as the level N tends to infinity.

The main references for this part are [AU97], [Con07], [KM85], and [May14].

5.1 The self-intersection number $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ of congruence subgroups

For the sequel, let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and denote by X_Γ/K the algebraic curve associated to the compact Riemann surface $X(\Gamma)$, with K a suitable field. Also, let $H \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be the subgroup determined by the function field of X_Γ/K.

Consider the natural map $f : X(\Gamma) \longrightarrow \mathbb{P}^1(\mathbb{C})$ given by the j-invariant and let $f^{-1}(1728) = \{P_1, \ldots, P_k\}$ and $f^{-1}(0) = \{Q_1, \ldots, Q_l\}$. Let us assume that $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ satisfies the following conditions:

(i) The ramification index of f at the points P_j is constant and similarly for the points Q_j;

(ii) Γ contains $\Gamma(N)$ for some composite odd square-free positive integer N such that $g_\Gamma \geq 2$;

(iii) the moduli problem $[\Gamma(N)]^{\mathrm{can}}/H$ on $(\mathrm{Ell}/\mathbb{Z}[\zeta_N]^{\det(H)})$ is representable by a scheme $\mathcal{X}_\Gamma/\mathbb{Z}[\zeta_N]^{\det(H)}$.

Condition (i) together with [AU97, Lemme 4.1.1, p. 61] imply that the canonical divisor of X_Γ/K is supported on the cusps. Therefore, there exist vertical divisors V_0^Γ and V_∞^Γ in $\mathrm{Z}^1(\mathcal{X}_\Gamma) \otimes \mathbb{Q}$ supported on the fibers $(\mathcal{X}_\Gamma)_\mathfrak{p}$ with $\mathfrak{p}|N$ satisfying (3.4). Consequently, the theorems of Manin–Drinfeld, Faltings–Hriljac, and Néron–Tate, can be applied in the same way as we did in the proof of Proposition 3.4.2, and therefore, we have

$$\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K} = \frac{2g_\Gamma(V_0^\Gamma, V_\infty^\Gamma)_{\mathrm{fin}} - (V_0^\Gamma, V_0^\Gamma)_{\mathrm{fin}} - (V_\infty^\Gamma, V_\infty^\Gamma)_{\mathrm{fin}}}{2(g_\Gamma - 1)} - 2g_\Gamma(g_\Gamma - 1)\sum_{\sigma: K \hookrightarrow \mathbb{C}} g^\Gamma_{\mathrm{can}}(0^\sigma, \infty^\sigma)$$

In the case of the congruence subgroups $\Gamma = \Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$, the divisors V_0^Γ and V_∞^Γ have been explicitly given (see (3.5) for the case $\Gamma = \Gamma(N)$); thus, the geometric part of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ can be calculated directly in each case (see Lemma 5.2.1). For the general case we propose the following method.

Description of the method

Let $\pi : \mathcal{X}_{\Gamma(N)} \longrightarrow \mathcal{X}_\Gamma$ be the natural morphism coming from the moduli interpretation of $\mathcal{X}_\Gamma$. By [Liu02, Theorem 2.12 (c), p. 398] we have the identity

$$(V_q, V_{\tilde{q}})_{\mathrm{fin}} = \frac{1}{\deg(\pi)}(\pi^* V_q, \pi^* V_{\tilde{q}})_{\mathrm{fin}}, \tag{5.1}$$

where $q, \tilde{q} \in \{0, \infty\}$. If $\mathfrak{p}|N$ and $C^\Gamma_{q,\mathfrak{p}}$ denotes the irreducible component of $(\mathcal{X}_\Gamma)_\mathfrak{p}$ intersected by the horizontal divisor H^Γ_q defined by q, then $\pi^* V^\Gamma_q$ is completely determined by the pullbacks $\pi^* C^\Gamma_{q,\mathfrak{p}}$.

Proposition 5.1.1. *Let N be a composite odd square-free positive integer greater than one and $\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N]$ the fine moduli scheme of $[\Gamma(N)]^{\mathrm{can}}$. Suppose that $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ such that $\mathfrak{p}|p$, for some $p|N$. Then the following assertions hold:*

(a) The set of irreducible components of the fiber $(\mathcal{X}_{\Gamma(N)})_\mathfrak{p}$ is parametrized by $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\backslash \mathrm{SL}_2(\mathbb{F}_p)$.

(b) The assignment $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto [c:d]$ *induces a bijection between* $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\backslash \mathrm{SL}_2(\mathbb{F}_p)$ *and* $\mathbb{P}^1(\mathbb{F}_p)$.

(c) Each irreducible component of the fiber $(\mathcal{X}_{\Gamma(N)})_{\mathfrak{p}}$ *has multiplicity one.*

As a result of the previous proposition, the set of irreducible components of $(\mathcal{X}_\Gamma)_{\mathfrak{p}}$ is parametrized by $\mathbb{P}^1(\mathbb{F}_p)/\overline{H}$, where $\overline{H}$ denotes the image of H in $\mathrm{GL}_2(\mathbb{F}_p)$, and therefore, the pullback $\pi^* C^{\Gamma}_{q,\mathfrak{p}}$ can be explicitly determined. This concludes the method.

Let us illustrate the previous method with an example.

Example 5.1.2. Recall that G_1 denotes the subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ given by matrices of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Let $\Gamma = \Gamma_1(N)$. Then the corresponding subgroup $H \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is given by $H = G_1$. Since $[\Gamma_1(N)]^{\mathrm{can}} = [\Gamma(N)]^{\mathrm{can}}/G_1$, we obtain

$$(\mathcal{X}_{\Gamma_1(N)})_{\mathfrak{p}} = (\mathcal{X}_{\Gamma_1(N)})_{\mathfrak{p}}/G_1.$$

By the right action of the subgroup $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \subset \mathrm{SL}_2(\mathbb{F}_p)$ on $\mathbb{P}^1(\mathbb{F}_p)$, we have that

$$\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\backslash \mathrm{SL}_2(\mathbb{F}_p)/\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) = \{\overline{[1:0]}, \overline{[0:1]}\},$$

because $[0:1]$ is fixed, whereas we have a free action on the set of points $[1:y]$ with $y \in \mathbb{P}^1(\mathbb{F}_p)$. Hence, these double cosets correspond to the two irreducible components in the fiber at $\mathfrak{p}$ in the description of $\mathcal{X}_{\Gamma_1(N)}/\mathbb{Z}[\zeta_N]$ given in Theorem 3.3.18 (see figure below).

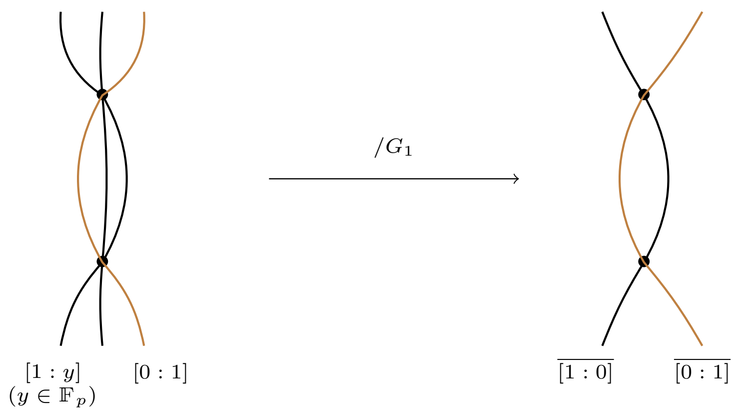

Figure 5.1: Administration of the irreducible components in the fiber at $\mathfrak{p}$.

Finally, since $C^{\Gamma}_{\infty,\mathfrak{p}}$ corresponds to $\overline{[0:1]}$, we can easily deduce the pullbacks

$$\pi^* C^{\Gamma_1(N)}_{\infty,\mathfrak{p}} = C^{\Gamma(N)}_{\infty,\mathfrak{p}} \qquad \text{and} \qquad \pi^* C^{\Gamma_1(N)}_{0,\mathfrak{p}} = \sum_{y \in \mathbb{F}_p} C^{\Gamma(N)}_{y,\mathfrak{p}}.$$

Now we proceed to prove Proposition 5.1.1. To do so, let us introduce two fundamental notions that will be needed: the Tate curve and the closed subscheme of cusps.

The Tate curve

In the following considerations, we follow [Con07]. By a *curve over a scheme* S we mean a morphism $C \longrightarrow S$ which is separated, flat, finitely presented, and of pure relative dimension 1, with all fibers non-empty.

Let $N > 1$ be an integer. The *standard N-gon over a scheme S* is the proper curve over S obtained from $\mathbb{P}^1_S \times \mathbb{Z}/N\mathbb{Z}$ by gluing the ∞-section of $\mathbb{P}^1_S \times \{i\}$ to the 0-section of $\mathbb{P}^1_S \times \{i+1\}$, for all $i \in \mathbb{Z}/N\mathbb{Z}$. A *Deligne–Rapoport semistable genus-1 curve over a scheme S* is a proper curve $f : E \longrightarrow S$ whose geometric fibers are connected and semistable with trivial relative dualizing sheaf.

Definition 5.1.3. A triple $(E, +, e)$ consisting of a Deligne–Rapoport semistable genus-1 curve E over S, an S-morphism $+ : E^{\mathrm{sm}} \times_S E \longrightarrow E$, and a section $e \in E^{\mathrm{sm}}[S]$, is called a *generalized elliptic curve over S* if the geometric fibers of E/S are either smooth curves of genus 1 or standard N-gons, and if the following conditions are satisfied:

(i) The restriction of the morphism $+$ to $E^{\mathrm{sm}} \times_S E^{\mathrm{sm}}$ defines a structure of commutative group scheme on E^{sm};

(ii) the morphism $+$ defines an action of E^{sm} on E;

(iii) on singular geometric fibers E_s, the translation action $y \longmapsto x + y$ by rational points $x \in E^{\mathrm{sm}}_s(k(s))$ acts on the graph of irreducible components via rotations.

Now we review the construction of the Tate curve given in [Con07, §2.5, p. 232]. Consider a noetherian ring R and let $J \subset R$ be an ideal such that it gives a separated and complete topology on R. Let $R\{\{T_1, \ldots, T_N\}\}$ denote the J-adic completion of $R[T_1, \ldots, T_N]$ and for $r \in R$, let us write $R\{\{1/r\}\}$ for the quotient $R\{\{T\}\}/(1 - rT)$. If $X = \mathrm{Spf}(R)$, then $X\{\{1/r\}\}$ stands for $\mathrm{Spf}(R\{\{1/r\}\})$. Here Spf refers to the formal spectrum.

For $i \in \mathbb{Z}/N\mathbb{Z}$, define the formal annulus

$$\Delta_i := \mathrm{Spf}\Big(\mathbb{Z}[\![q^{1/N}]\!]\{\{X_i, Y_i\}\}/(X_iY_i - q^{1/N})\Big).$$

Let Δ_i^+ and Δ_i^- be the open formal subschemes given by $\Delta_i^+ := \Delta_i\{\{1/X_i\}\}$ and $\Delta_i^- := \Delta_i\{\{1/Y_i\}\}$. By gluing Δ_i^- with Δ_{i+1}^+ via $Y_i = X_{i+1}$ and $X_i = Y_{i+1}$, for all $i \in \mathbb{Z}/N\mathbb{Z}$ we obtain a formal scheme which is algebraizable, i.e., it is obtained by completing a single noetherian scheme $\mathscr{T}$ along some closed subscheme ([Har77, Example 9.3.2, p. 195]).

Definition 5.1.4. The *Tate curve* $\mathrm{Tate}/\mathrm{Spec}(\mathbb{Z}[\![q^{1/N}]\!])$ is the $\mathrm{Spec}(\mathbb{Z}[\![q^{1/N}]\!])$-scheme $\mathscr{T}$ given in the previous construction.

In the following proposition we state the main properties of the Tate curve.

Proposition 5.1.5. *The Tate curve is a generalized elliptic curve with N-gon geometric fibers over the locus $q^{1/N} = 0$ and smooth outside this locus. Over $\mathbb{Z}(\!(q)\!)$, the Tate curve is an elliptic curve in the sense of Definition 3.3.1. Furthermore, there is a natural isomorphism of $\mathbb{Z}[\![q^{1/N}]\!]$-schemes*

$$\mathrm{Tate}^{\mathrm{sm}}[N] \simeq \mu_N \times \mathbb{Z}/N\mathbb{Z},$$

where $\mathrm{Tate}^{\mathrm{sm}}[N]$ *stands for the set of N-torsion points of the smooth part of the Tate curve.*

Proof. For the proof we refer the reader to [Con07, p. 233–234]. □

The closed subscheme of cusps

Let R be an excellent noetherian regular ring. Suppose that $\mathcal{P}$ is a relatively representable moduli problem which is affine and finite over (Ell/R). Let $M(\mathcal{P})$ denote the coarse moduli scheme of $\mathcal{P}$, which is a scheme over $\mathbb{A}^1_R := \mathrm{Spec}(R[j])$ (see [KM85, (8.1), p. 224]). If in addition there exists a monic polynomial $f(j) \in R[j]$ such that on the open set $U \subset \mathbb{A}^1_R$ where f is invertible, the scheme $M(\mathcal{P}) \times_{R[j]} U$ is normal, then $M(\mathcal{P})$ extends by normalization to a scheme $\overline{M}(\mathcal{P})$ over $\mathbb{P}^1_R$. Thus, we can define

$$\mathrm{Cusps}(\mathcal{P}) := \left(\overline{M}(\mathcal{P}) \setminus M(\mathcal{P})\right)^{\mathrm{red}}.$$

Let $\widehat{\mathrm{Cusps}}(\mathcal{P})$ be the formal completion of $\overline{M}(\mathcal{P})$ along $\mathrm{Cusps}(\mathcal{P})$. Katz–Mazur proved in [KM85, Theorem 8.11.10, p. 269] that $\widehat{\mathrm{Cusps}}(\mathcal{P})$ is the (relative) normalization of $R[\![q]\!]$ in the finite normal $R(\!(q)\!)$-scheme $\left(\mathcal{P}_{\mathrm{Tate}/R(\!(q)\!)}\right)/\{\pm 1\}$. Thus, loosely speaking, cusps corresponds to $\mathcal{P}$-level structures on the Tate curve.

Now we are ready to prove the Proposition 5.1.1.

Proof of Proposition 5.1.1. One the one hand, by [KM85, Theorem 10.8.2,

p. 299], we have

$$\widehat{\mathrm{Cusps}}([\Gamma(N)]^{\mathrm{can}}) \simeq ([\Gamma(N)]^{\mathrm{can}}_{\mathrm{Tate}/\mathbb{Z}((q))})/\{\pm 1\}$$
$$\simeq \bigsqcup_{\{\pm 1\}G_1\backslash \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[\zeta_N][\![q^{1/N}]\!]). \tag{5.2}$$

On the other hand, by the cusps/components label given by [KM85, (10.6), p. 295], the irreducible components of $(\mathcal{X}_{\Gamma(N)})_{\mathfrak{p}}$, where $\mathfrak{p}|p$ with $p|N$ are parametrized by $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\backslash \mathrm{SL}_2(\mathbb{F}_p)$. The result follows. $\square$

Complement of Chapter 4

We end this section with an application of the new notions which complements Chapter 4. In Section 4.1 we claimed that for the computation of the analytic contribution of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$, with $\Gamma = \Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$, it suffices to consider only (4.2). Here, we justify the case $\Gamma = \Gamma(N)$.

Let us first recall and clarify the settings of Chapter 4: We have cusps $0, \infty \in C_\Gamma$, rational points $0, \infty \in X_\Gamma(\mathbb{Q}(\zeta_N))$ that corresponds to the previous cusps (note the abuse of notation), an embedding $\sigma : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$, and points $0^\sigma, \infty^\sigma \in X_\sigma$ that corresponds to $0, \infty \in X_\Gamma(\mathbb{Q}(\zeta_N))$ after we base change $X_\Gamma/\mathbb{Q}(\zeta_N)$ to $\mathbb{C}$ via σ.

Proposition 5.1.6. *Let $\Gamma = \Gamma(N)$. Suppose that $\sigma : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$ is an embedding such that $\sigma(\zeta_N) = e^{2\pi i v/N}$ with $vv' \equiv 1 \bmod N$. Then there exists an isomorphism*

$$\iota_\sigma : X_\sigma \xrightarrow{\simeq} X(\Gamma)$$

such that $\iota_\sigma(0^\sigma) = [1 : v']$ and $\iota_\sigma(\infty^\sigma) = [1 : 0]$, where $v' \in (\mathbb{Z}/N\mathbb{Z})^\times$.

In order to find an isomorphism ι_σ explicitly, we need a moduli interpretation for the cusps of $\mathcal{X}_\Gamma/\mathbb{Z}[\zeta_N]$. This will be done by extending the $\Gamma(N)$-structures of Chapter 3 to generalized elliptic curves.

Definition 5.1.7. Let E/S be a generalized elliptic curve over a scheme S. An *extended* $\Gamma(N)$*-structure on* E/S is an ordered pair (P, Q) with $P, Q \in E^{\mathrm{sm}}[N](S)$ such that the Cartier divisor

$$\sum_{(a,b)\in(\mathbb{Z}/N\mathbb{Z})^2} [aP + bQ]$$

meets all irreducible components of all geometric fibers and equals $E^{\mathrm{sm}}[N]$.

If we consider the contravariant functor on the category of generalized elliptic curves which assigns to each object of the category the set of $\Gamma(N)$-structures on it, then for N sufficiently large, this functor is representable by the scheme $\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N]$ (see [Con07, Remark 4.2.2, p. 259]).

For the rest of the section, we consider the scheme $\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N]$ as classifying extended $\Gamma(N)$-structures on generalized elliptic curves. Thus, the closed subscheme $\mathrm{Cusps}([\Gamma(N)])$ corresponds to the extended $[\Gamma(N)]$-structures of the standard N-gon.

Let us consider now the Tate curve together with an extended $\Gamma(N)$-structure on it. If we base change the Tate curve via $\mathbb{Z}[\![q^{1/N}]\!] \longrightarrow \mathbb{Q}[\![q^{1/N}]\!][\zeta_N]$, the points $P = q^{1/N}$ and $Q = \zeta_N$ of $\mathrm{Tate}(\mathbb{Q}(\!(q^{1/N})\!)[\zeta_N])$ give a canonical basis for $\mathrm{Tate}^{\mathrm{sm}}[N]$. By specializing to $q^{1/N} = 0$, we obtain a curve over $\mathbb{Q}(\zeta_N)$ denoted by $\mathrm{Tate}_0/\mathbb{Q}(\zeta_N)$ such that

$$(\mathrm{Tate}_0)^{\mathrm{sm}} = \mathbb{G}_{\mathrm{m}} \times \mathbb{Z}/N\mathbb{Z},$$

$$\mathrm{Tate}_0[N] = \mathbb{Z}/N\mathbb{Z} \times \mu_N,$$

with a canonical basis of $\mathrm{Tate}_0[N]$ given by the points

$$P_0 = (1, 1), \;\; Q_0 = (0, \zeta_N).$$

Suppose that $\mathrm{Tate}_{0,\mathbb{C}}$ is the resulting curve by the base change of $\mathrm{Tate}_0/\mathbb{Q}(\zeta_N)$ to $\mathbb{C}$. Then an extended $\Gamma(N)$-structure on $\mathrm{Tate}_{0,\mathbb{C}}[N]$ is performed by an isomorphism

$$\begin{aligned} \varphi_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} : (\mathbb{Z}/N\mathbb{Z})^2 &\longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mu_N \\ (1,0) &\longmapsto (a, \zeta_N^b) \\ (0,1) &\longmapsto (c, \zeta_N^d), \end{aligned}$$

where μ_N denotes the group of N-th roots of unity and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Note that

$$\varphi_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \quad \text{and} \quad \varphi_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$$

correspond to the canonical basis (P_0, Q_0) and $(-Q_0, P_0)$ of $\mathrm{Tate}_{0,\mathbb{C}}[N]$, respectively. They represent the cusps ∞ and 0 of $\Gamma\backslash\mathbb{H}$, respectively.

Proof of Proposition 5.1.6. First of all, note that any embedding σ is determined by the assignment

$$\zeta_N \longmapsto e^{2\pi i v/N}$$

for some $v \in (\mathbb{Z}/N\mathbb{Z})^\times$. Secondly, let $(E; P, Q) \in X_\Gamma/\mathbb{Q}(\zeta_N)$ be a triple consisting of a generalized elliptic curve E over $\mathbb{Q}(\zeta_N)$ and an extended $\Gamma(N)$-structure on E given by a basis $P, Q \in E[N](\mathbb{Q}(\zeta_N))$ such that $e_N(P, Q) = \zeta_N$; here, $e_N(\cdot, \cdot)$ denotes the Weil pairing. The base change of this triple via the embedding σ gives $({}^\sigma E; {}^\sigma P, {}^\sigma Q) \in X_\sigma/\mathbb{C}$, where ${}^\sigma E$ is a generalized elliptic curve over $\mathbb{C}$ and $({}^\sigma P, {}^\sigma Q)$ is a basis of ${}^\sigma E[N]$ satisfying

$$e_N({}^\sigma P, {}^\sigma Q) = {}^\sigma \zeta_N = e^{2\pi i v/N}.$$

Thirdly, we define the isomorphism ι_σ as follows:

$$\iota_\sigma({}^\sigma E; {}^\sigma P, {}^\sigma Q) := \Gamma(N)\tau$$

for some $\tau \in \mathbb{H}$ satisfying the following property: There is a $\gamma_v \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau' \in \mathbb{H}$ such that $\tau = \gamma_v \tau'$ and $({}^\sigma E^{\mathrm{sm}}, v \cdot {}^\sigma P + {}^\sigma Q, v' \cdot {}^\sigma Q)$ is isomorphic to the limit of $(\mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z}); \tau/N, 1/N)$, as $\mathrm{Im}(\tau') \to \infty$; here, $v' \in (\mathbb{Z}/N\mathbb{Z})^\times$ is such that $vv' \equiv 1 \bmod N$.

Now, we verify that ι_σ satisfies the desired condition on the cusp 0^σ. The triple $(\mathrm{Tate}_0; -Q_0, P_0)$ corresponds to $0 \in X_\Gamma/\mathbb{Q}(\zeta_N) = (\mathcal{X}_\Gamma)_\eta$. By base change via σ, we have now $(\mathrm{Tate}_{0,\mathbb{C}}; \zeta_N^{-v}, X^{1/N})$. Note that

$$(\mathrm{Tate}_{0,\mathbb{C}}^{\mathrm{sm}}; \zeta_N^{-v} X, X^{v'/N}) \simeq (\mathbb{C}^\times/(e^{2\pi i \tau'})^{\mathbb{Z}}; e^{2\pi i(\tau'-v)/N}, e^{2\pi i v'\tau'/N})$$

for some $\tau' \in \mathbb{H}$. Furthermore, using the inverse of the exponential map, we have

$$(\mathrm{Tate}_{0,\mathbb{C}}^{\mathrm{sm}}; \zeta_N^{-v} X, X^{v'/N}) \simeq (\mathbb{C}/(\mathbb{Z}\tau' + \mathbb{Z}); (\tau' - v)/N, v'\tau'/N),$$

as $\mathrm{Im}(\tau') \to \infty$. Thus, it remains to find $\gamma_v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau = \gamma_v \tau'$ and

$$(\mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z}); \tau/N, 1/N) \simeq (\mathbb{C}/(\mathbb{Z}\tau' + \mathbb{Z}); (\tau' - v)/N, v'\tau'/N).$$

It can be easily verified that such a matrix must satisfy $\gamma_v \equiv \left(\begin{smallmatrix} 1 & -v \\ v' & 0 \end{smallmatrix}\right) \bmod N$. Consequently, as $\mathrm{Im}(\tau') \to \infty$, so $\tau \to a/c = [1 : v']$. Then $0^{\sigma_v} = [1 : v']$, where $vv' \equiv 1 \bmod N$. A similar analysis can be done to prove $\iota_\sigma(\infty^\sigma) = [1 : 0]$. This concludes the proof. □

5.2 Asymptotic analysis of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$

In the following lemma we determine the geometric contribution of $\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathcal{O}_K}$ given in Proposition 3.4.2.

Lemma 5.2.1. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$ with N satisfying the conditions for the existence of the fine moduli scheme $\mathcal{X}_\Gamma/\mathcal{O}_K$ given in Section 3.3 such that $g_\Gamma > 1$. Then the following assertions hold:*

(a) If $\Gamma = \Gamma_0(N)$, then we have

$$\frac{2g_\Gamma (V_0, V_\infty)_{\mathrm{fin}} - (V_0, V_0)_{\mathrm{fin}} - (V_\infty, V_\infty)_{\mathrm{fin}}}{2(g_\Gamma - 1)} = 12(g_\Gamma + 1)\frac{g_\Gamma - 1}{\sigma(N)} \sum_{p|N} \frac{p+1}{p-1} \log(p).$$

(b) If $\Gamma = \Gamma_1(N)$, then we have

$$\frac{2g_\Gamma (V_0, V_\infty)_{\mathrm{fin}} - (V_0, V_0)_{\mathrm{fin}} - (V_\infty, V_\infty)_{\mathrm{fin}}}{2(g_\Gamma - 1)} = 24(g_\Gamma + 1)\frac{g_\Gamma - 1}{\sigma(N)} \sum_{p|N} \frac{p+1}{p-1} \log(p).$$

(c) If $\Gamma = \Gamma(N)$, then we have

$$\frac{2g_\Gamma (V_0, V_\infty)_{\mathrm{fin}} - (V_0, V_0)_{\mathrm{fin}} - (V_\infty, V_\infty)_{\mathrm{fin}}}{2(g_\Gamma - 1)} = 4\varphi(N)\frac{N-6}{N}\left(g_\Gamma \sum_{p|N} \frac{p}{p^2-1} \log(p) + \sum_{p|N} \frac{p^2}{p^2-1} \log(p) \right).$$

Proof. For the proof of parts (a) and (b), we refer the reader to [AU97, Proposition 4.2.1, p. 63] and [May14, 7.5 Proposition, p. 38], respectively. For the proof of part (c), recall that

$$V_0 = -\sum_{\mathfrak{p}|N} \frac{2(g_\Gamma - 1)}{r_\mathfrak{p} s_\mathfrak{p}} C_{0,\mathfrak{p}},$$

$$V_\infty = -\sum_{\mathfrak{p}|N} \frac{2(g_\Gamma - 1)}{r_\mathfrak{p} s_\mathfrak{p}} C_{\infty,\mathfrak{p}}$$

(see (3.5) in the proof of Lemma 3.4.1), where $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ is such that $\mathfrak{p}|p$ with $p|N$ a prime number, $r_\mathfrak{p} := p+1$, and $s_\mathfrak{p}$ is given by

$$s_\mathfrak{p} = \frac{p-1}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N/p)].$$

Furthermore, $C_{0,\mathfrak{p}}$ resp. $C_{\infty,\mathfrak{p}}$ denotes the irreducible component in the fiber at $\mathfrak{p}$ intersected by the cusp 0 and ∞, respectively. Using the fact that N is square-free and also the identity

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(M)] = M^3 \prod_{q|M} \left(1 - \frac{1}{q^2}\right),$$

we obtain the following formula for $s_{\mathfrak{p}}$:

$$s_{\mathfrak{p}} = \frac{N}{24p(p+1)} \prod_{q|N} (q^2 - 1), \tag{5.3}$$

where $\mathfrak{p}|p$ and the product runs over all primes dividing N. Now, we claim that

$$(V_0, V_\infty)_{\text{fin}} = 4(g_\Gamma - 1)\left(\frac{N-6}{N}\right)\varphi(N) \sum_{p|N} \frac{p}{p^2-1} \log(p). \tag{5.4}$$

Indeed, note that

$$\begin{aligned}
(V_0, V_\infty)_{\text{fin}} &= \sum_{\mathfrak{p}|N} \sum_{\mathfrak{q}|N} \frac{4(g_\Gamma - 1)^2}{r_{\mathfrak{p}} r_{\mathfrak{q}} s_{\mathfrak{p}} s_{\mathfrak{q}}} (C_{0,\mathfrak{p}}, C_{\infty,\mathfrak{q}})_{\text{fin}} \\
&= \sum_{\mathfrak{p}|N} \frac{4(g_\Gamma - 1)^2}{r_{\mathfrak{p}}^2 s_{\mathfrak{p}}^2} (C_{0,\mathfrak{p}}, C_{\infty,\mathfrak{p}})_{\text{fin}} \\
&= \sum_{\mathfrak{p}|N} \frac{4(g_\Gamma - 1)^2}{r_{\mathfrak{p}}^2 s_{\mathfrak{p}}^2} \Big(s_{\mathfrak{p}} \log(\#k(\mathfrak{p}))\Big),
\end{aligned}$$

where in the second equality we used $(C_{0,\mathfrak{p}}, C_{\infty,\mathfrak{q}})_{\text{fin}} = 0$ provided that $\mathfrak{p} \neq \mathfrak{q}$ and in the last equality we used (3.6) in the proof of the Lemma 3.4.1. Moreover, we have

$$\begin{aligned}
(V_0, V_\infty)_{\text{fin}} &= \sum_{\mathfrak{p}|N} \frac{4(g_\Gamma - 1)^2}{r_{\mathfrak{p}}^2 s_{\mathfrak{p}}} \log(\#k(\mathfrak{p})) \\
&= \sum_{p|N} \sum_{\mathfrak{p}|p} \frac{4(g_\Gamma - 1)^2}{r_{\mathfrak{p}}^2 s_{\mathfrak{p}}} \log(\#k(\mathfrak{p})) \\
&= \frac{4(g_\Gamma - 1)}{N} \frac{24(g_\Gamma - 1)}{\prod_{q|N}(q^2-1)} \sum_{p|N} \frac{p}{p+1} \sum_{\mathfrak{p}|p} \log(\#k(\mathfrak{p})),
\end{aligned}$$

where in the last equality we used (5.3). The claim follows because we have

$$\frac{24(g_\Gamma - 1)}{\prod_{q|N}(q^2-1)} = N - 6,$$

$$\sum_{\mathfrak{p}|p} \log(\#k(\mathfrak{p})) = \varphi(N/p) \log(p) = \frac{\varphi(N)}{p-1} \log(p).$$

Similarly, the following identities can be proved

$$(V_\infty, V_\infty)_{\text{fin}} = (V_0, V_0)_{\text{fin}} = -4(g_\Gamma - 1)\left(\frac{N-6}{N}\right)\varphi(N) \sum_{p|N} \frac{p^2}{p^2-1} \log(p) \tag{5.5}$$

Consequently, we have

$$\frac{2g_{\Gamma}(V_0,V_\infty)_{\mathrm{fin}}-(V_0,V_0)_{\mathrm{fin}}-(V_\infty,V_\infty)_{\mathrm{fin}}}{2(g_{\Gamma}-1)}=\frac{g_{\Gamma}(V_0,V_\infty)_{\mathrm{fin}}-(V_0,V_0)_{\mathrm{fin}}}{g_{\Gamma}-1},$$

and using (5.4) and (5.5), we finally obtain

$$\frac{g_{\Gamma}(V_0,V_\infty)_{\mathrm{fin}}-(V_0,V_0)_{\mathrm{fin}}}{g_{\Gamma}-1}=\frac{4(N-6)}{N}\varphi(N)\sum_{p|N}\frac{g_{\Gamma}+p}{p^2-1}p\log(p).$$

This concludes the proof. □

Before we proceed to analyze the asymptotic behaviour of $\overline{\omega}^2_{\mathcal{X}_{\Gamma}/\mathcal{O}_K}/[K:\mathbb{Q}]$ as $N\to\infty$, the following lemma will be useful.

Lemma 5.2.2. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$. Denote by $Z_\Gamma(s)$ the Selberg zeta function. Then for all $\varepsilon>0$, the following holds*

$$\lim_{s\to1}\left(\frac{Z'_\Gamma}{Z_\Gamma}(s)-\frac{1}{s-1}\right)=O_\varepsilon(N^\varepsilon).$$

Proof. For the proof we refer the reader to [JK06a, p. 27]. □

Theorem 5.2.3. *Let Γ be one of the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$ with N satisfying the conditions for the existence of the fine moduli scheme $\mathcal{X}_\Gamma/\mathcal{O}_K$ given in Section 3.3 such that $g_{\Gamma}>1$. Then, as $N\to\infty$, the following assertions hold:*

(a) If $\Gamma=\Gamma_0(N)$, then we have

$$\frac{1}{\varphi(N)}\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathbb{Z}}=3g_{\Gamma_0(N)}\log(N)+o(g_{\Gamma_0(N)}\log(N)).$$

(b) If $\Gamma=\Gamma_1(N)$, then we have

$$\frac{1}{\varphi(N)}\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathbb{Z}[\zeta_N]}=3g_{\Gamma_1(N)}\log(N)+o(g_{\Gamma_1(N)}\log(N)).$$

(c) If $\Gamma=\Gamma(N)$, then we have

$$\frac{1}{\varphi(N)}\overline{\omega}^2_{\mathcal{X}_\Gamma/\mathbb{Z}[\zeta_N]}=4g_{\Gamma(N)}\log(N)+o(g_{\Gamma(N)}\log(N)).$$

Proof. We prove part (c) since it is not in the literature. Parts (a) and (b) can be done similarly or the reader can consult [AU97] and [May14], respectively.

Let us first consider the analytic contribution $\mathcal{A}(N)$ of $\overline{\omega}^2_{\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N]}/\varphi(N)$, namely, we have

$$\mathcal{A}(N) \coloneqq -\frac{2g_{\Gamma(N)}(g_{\Gamma(N)}-1)}{\varphi(N)} \sum_{\sigma:\mathbb{Q}(\zeta_N)\hookrightarrow\mathbb{C}} g_{\mathrm{can}}^{\Gamma(N)}(0^\sigma,\infty^\sigma)$$

(see Proposition 3.4.2). Recall that each embedding $\sigma : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$ is determined by $\zeta_N \longmapsto \zeta_N^v$, where $v \in (\mathbb{Z}/N\mathbb{Z})^\times$; then by Proposition 5.1.6 we have

$$\frac{1}{\varphi(N)} \sum_{\sigma:\mathbb{Q}(\zeta_N)\hookrightarrow\mathbb{C}} g_{\mathrm{can}}^{\Gamma(N)}(0^\sigma,\infty^\sigma) = \frac{1}{\varphi(N)} \sum_{v\in(\mathbb{Z}/N\mathbb{Z})^\times} g_{\mathrm{can}}^{\Gamma(N)}(0_v,\infty).$$

Now recall (4.3) from Theorem 4.1.1, namely, we have

$$g_{\mathrm{can}}^{\Gamma(N)}(0_v,\infty) = 4\pi\mathscr{C}_{0_v\infty}^{\Gamma(N)} + \frac{4\pi}{v_{\Gamma(N)}} - 8\pi\mathscr{R}_\infty^{\Gamma(N)} + O\left(\frac{1}{g_{\Gamma(N)}}\right); \tag{5.6}$$

here, the scattering constants $\mathscr{C}_{0_v\infty}^{\Gamma(N)}$ for $v \in (\mathbb{Z}/N\mathbb{Z})^\times$ are given by (see Theorem 2.2.12)

$$\begin{aligned}\mathscr{C}_{0_v\infty}^{\Gamma(N)} &= 2v_{\Gamma(N)}^{-1}\left(\mathscr{C} + \frac{1}{2}\sum_{p|N}\frac{1+2p-p^2}{p^2-1}\log(p) - \frac{1}{2}\log(N)\right) + \frac{2\pi}{N^2\varphi(N)}\kappa_{N,v} \\ &= 2v_{\Gamma(N)}^{-1}\left(\mathscr{C} + \sum_{p|N}\frac{p\log(p)}{p^2-1} - \log(N)\right) + \frac{2\pi}{N^2\varphi(N)}\kappa_{N,v}.\end{aligned}$$

Furthermore, the constant $\mathscr{R}_\infty^{\Gamma(N)}$ is given in part (c) of Theorem 4.1.1 and it is equal to

$$\begin{aligned}\mathscr{R}_\infty^{\Gamma(N)} = &-\frac{v_{\Gamma(N)}^{-1}}{2g_{\Gamma(N)}}\lim_{s\to1}\left(\frac{Z'_{\Gamma(N)}}{Z_{\Gamma(N)}}(s) - \frac{1}{s-1}\right) + \frac{1-\log(4\pi)}{4\pi g_{\Gamma(N)}} + \frac{\mathscr{C}_{\infty\infty}^{\Gamma(N)}}{g_{\Gamma(N)}} \\ &+ \frac{12}{\pi g_{\Gamma(N)}N}\left[C_1 - \mathscr{C} - \frac{\gamma_{\mathrm{EM}}}{2} + \log(N)\right],\end{aligned}$$

where (see Theorem 2.2.12)

$$\begin{aligned}\mathscr{C}_{\infty\infty}^{\Gamma(N)} &= 2v_{\Gamma(N)}^{-1}\left(\mathscr{C} - \sum_{p|N}\frac{p^2\log(p)}{p^2-1} - \log(N)\right) \\ &= 2v_{\Gamma(N)}^{-1}\left(\mathscr{C} - \sum_{p|N}\frac{\log(p)}{p^2-1} - 2\log(N)\right).\end{aligned}$$

In what follows, we write $g = g_{\Gamma(N)}$ to simplify the notation.

From the first two terms in the identity (5.6), we obtain

$$4\pi\mathscr{C}^{\Gamma(N)}_{0_v\infty}+\frac{4\pi}{v_{\Gamma(N)}}=\frac{4\pi}{v_{\Gamma(N)}}+\frac{8\pi}{v_{\Gamma(N)}}\left(\mathscr{C}+\sum_{p|N}\frac{p\log(p)}{p^2-1}-\log(N)\right)+\frac{8\pi^2}{N^2\varphi(N)}\kappa_{N,v}$$
$$=\frac{8\pi}{v_{\Gamma(N)}}\left(\frac{1}{2}+\mathscr{C}+\sum_{p|N}\frac{p\log(p)}{p^2-1}\right)-\frac{8\pi}{v_{\Gamma(N)}}\log(N)+\frac{8\pi^2}{N^2\varphi(N)}\kappa_{N,v}.$$

Similarly, if we consider the third term in (5.6), omitting the sign, we have

$$8\pi\mathscr{R}^{\Gamma(N)}_{\infty}=-\frac{4\pi}{gv_{\Gamma(N)}}\lim_{s\to1}\left(\frac{Z'_{\Gamma(N)}}{Z_{\Gamma(N)}}(s)-\frac{1}{s-1}\right)+\frac{2}{g}(1-\log(4\pi))$$
$$+\frac{16\pi}{gv_{\Gamma(N)}}\left(\mathscr{C}-\sum_{p|N}\frac{\log(p)}{p^2-1}-2\log(N)\right)+\frac{96}{gN}\left[C_1-\mathscr{C}-\frac{\gamma_{\mathrm{EM}}}{2}+\log(N)\right].$$

Now, we put

$$-\frac{1}{\varphi(N)}\sum_{v\in(\mathbb{Z}/N\mathbb{Z})^\times}g^{\Gamma(N)}_{\mathrm{can}}(0_v,\infty)=m(N)+e_1(N)+e_2(N)+e_3(N)+e_4(N),$$

where $m(N)$, $e_1(N)$, $e_2(N)$, $e_3(N)$, and $e_4(N)$ are given by

$$\begin{aligned}
m(N)&:=\frac{8\pi}{v_{\Gamma(N)}}\left(1-\frac{4}{g}\right)\log(N),\\
e_1(N)&:=-\frac{8\pi^2}{(N\varphi(N))^2}\sum_{v\in(\mathbb{Z}/N\mathbb{Z})^\times}\kappa_{N,v},\\
e_2(N)&:=\frac{16\pi\mathscr{C}}{gv_{\Gamma(N)}}-\frac{4\pi}{gv_{\Gamma(N)}}\lim_{s\to1}\left(\frac{Z'_{\Gamma(N)}}{Z_{\Gamma(N)}}(s)-\frac{1}{s-1}\right),\\
e_3(N)&:=-\frac{4\pi}{v_{\Gamma(N)}}\left(2\mathscr{C}+1+2\sum_{p|N}\frac{p\log(p)}{p^2-1}+\frac{4}{g}\sum_{p|N}\frac{\log(p)}{p^2-1}\right),\\
e_4(N)&:=\frac{2}{g}(1-\log(4\pi))+\frac{96}{Ng}\left[C_1-\mathscr{C}-\frac{\gamma_{\mathrm{EM}}}{2}+\log(N)\right].
\end{aligned}$$

On the one hand, observe that $e_1(N)=0$. Indeed, by (2.7) of Notation 2.2.10, we know that

$$\kappa_{N,v}=\sum_{\substack{\chi\neq\chi_0\\ \text{even}}}\overline{\chi}(v')\frac{L(1,\chi)}{L(2,\chi)}=\sum_{\substack{\chi\neq\chi_0\\ \text{even}}}\chi(v)\frac{L(1,\chi)}{L(2,\chi)},$$

where in the second equality we used $\overline{\chi}(v')=\chi(v)$ (see (D.1) of Appendix D).

Then we have the following

$$
\begin{aligned}
\frac{1}{\varphi(N)} \sum_{v\in(\mathbb{Z}/N\mathbb{Z})^\times} \kappa_{N,v} &= \frac{1}{\varphi(N)} \sum_{v\in(\mathbb{Z}/N\mathbb{Z})^\times} \sum_{\substack{\chi\neq\chi_0\\ \text{even}}} \chi(v) \frac{L(1,\chi)}{L(2,\chi)} \\
&= \sum_{\substack{\chi\neq\chi_0\\ \text{even}}} \left(\frac{1}{\varphi(N)} \sum_{v\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(v) \right) \frac{L(1,\chi)}{L(2,\chi)} \\
&= 0,
\end{aligned}
$$

where in the last equality we used the orthogonality relations of Appendix D. On the other hand, note that $2g(g-1)e_4(N) = o(g\log(N))$. Indeed, we have

$$
\begin{aligned}
\frac{2g(g-1)e_4(N)}{g\log(N)} = \frac{1-\log(4\pi)}{\log(N)} \left(\frac{4(g-1)}{g} \right) \\
+ \frac{96}{N} \left(\frac{2(g-1)}{g} \right) \left(\frac{C_1 - \mathscr{C} - (\gamma_{\mathrm{EM}}/2) + \log(N)}{\log(N)} \right).
\end{aligned}
$$

The result follows since, on the right hand side, all the terms in parenthesis are bounded, whereas the terms outside the parenthesis go to zero as N tends to infinity.

Now we proceed to analyze the behaviour of $2g(g-1)m(N)$, $2g(g-1)e_2(N)$, and $2g(g-1)e_3(N)$ as $N \to \infty$. First of all, we claim that

$$
2g(g-1)m(N) = 4g\log(N) + o(g\log(N)), \tag{5.7}
$$

as $N \to \infty$. Indeed, we have

$$
\begin{aligned}
2g(g-1)m(N) &= 2g(g-1) \left[\frac{8\pi}{v_{\Gamma(N)}} \left(1 - \frac{4}{g}\right) \log(N) \right] \\
&= 4g\log(N) \left(\frac{4\pi(g-1)}{v_{\Gamma(N)}} \right) \left(1 - \frac{4}{g}\right) \\
&= 4g\log(N) \left(1 - \frac{6}{N}\right) \left(1 - \frac{4}{g}\right) \\
&= 4g\log(N) + 4g\log(N) \left(\frac{24}{gN} - \frac{6}{N} - \frac{4}{g} \right),
\end{aligned}
$$

where in the third equality we used part (c) of Lemma 2.2.3. The result follows since the term

$$
\frac{24}{gN} - \frac{6}{N} - \frac{4}{g}
$$

goes to zero as N tends to infinity.

Secondly, we claim that

$$2g(g-1)e_2(N) = o(g\log(N)), \tag{5.8}$$

as $N \to \infty$. Indeed, we have

$$\begin{aligned}
2g(g-1)e_2(N) &= 2g(g-1)\left[\frac{16\pi\mathscr{C}}{gv_{\Gamma(N)}} - \frac{4\pi}{gv_{\Gamma(N)}}\lim_{s\to 1}\left(\frac{Z'_{\Gamma(N)}}{Z_{\Gamma(N)}}(s) - \frac{1}{s-1}\right)\right] \\
&= \frac{4\pi(g-1)}{v_{\Gamma(N)}}\left[8\mathscr{C} - 2\lim_{s\to 1}\left(\frac{Z'_{\Gamma(N)}}{Z_{\Gamma(N)}}(s) - \frac{1}{s-1}\right)\right] \\
&= \left(1 - \frac{6}{N}\right)\left[8\mathscr{C} - 2\lim_{s\to 1}\left(\frac{Z'_{\Gamma(N)}}{Z_{\Gamma(N)}}(s) - \frac{1}{s-1}\right)\right] \\
&= O_\varepsilon(N^\varepsilon)
\end{aligned}$$

with $\varepsilon > 0$, where in the third equality we used part (c) of Lemma 2.2.3 and in the last equality we used Lemma 5.2.2. In particular, if we divide $2g(g-1)e_2(N)$ by $g\log(N)$, the resulting expression goes to zero as N goes to infinity for a sufficiently small ε, i.e., we have $2g(g-1)e_2(N) = o(g\log(N))$.

Thirdly, we claim that

$$2g(g-1)e_3(N) = o(g\log(N)), \tag{5.9}$$

as $N \to \infty$. Indeed, we have

$$\begin{aligned}
2g(g-1)e_3(N) &= 2g(g-1)\left[-\frac{4\pi}{v_{\Gamma(N)}}\left(2\mathscr{C} + 1 + 2\sum_{p|N}\frac{p\log(p)}{p^2-1} + \frac{4}{g}\sum_{p|N}\frac{\log(p)}{p^2-1}\right)\right] \\
&= -2g\left(\frac{4\pi(g-1)}{v}\right)\left[2\mathscr{C} + 1 + 2\sum_{p|N}\frac{p\log(p)}{p^2-1} + \frac{4}{g}\sum_{p|N}\frac{\log(p)}{p^2-1}\right].
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{p|N}\frac{p\log(p)}{p^2-1} &= \frac{1}{2}\left(\sum_{p|N}\frac{\log(p)}{p+1} + \sum_{p|N}\frac{\log(p)}{p-1}\right), \\
\sum_{p|N}\frac{\log(p)}{p^2-1} &= \frac{1}{2}\left(\sum_{p|N}\frac{\log(p)}{p-1} - \sum_{p|N}\frac{\log(p)}{p+1}\right),
\end{aligned}$$

then we have

$$\begin{aligned}
2g(g-1)e_3(N) &= -2g\left(\frac{4\pi(g-1)}{v}\right)\left[2\mathscr{C} + 1 + 2\sum_{p|N}\frac{p\log(p)}{p^2-1} + \frac{4}{g}\sum_{p|N}\frac{\log(p)}{p^2-1}\right] \\
&= -2g\left(1 - \frac{6}{N}\right)\left[2\mathscr{C} + 1 + h_1(N) + h_2(N)\right],
\end{aligned}$$

where

$$h_1(N) = \left(1 - \frac{2}{g}\right) \sum_{p|N} \frac{\log(p)}{p+1}$$

$$h_2(N) = \left(1 + \frac{2}{g}\right) \sum_{p|N} \frac{\log(p)}{p-1}.$$

Since we have

$$\sum_{p|N} \frac{\log(p)}{p-1} < 2 \sum_{p|N} \frac{\log(p)}{p}, \tag{5.10}$$

$$\sum_{p|N} \frac{\log(p)}{p+1} < 2 \sum_{p|N} \frac{\log(p)}{p}, \tag{5.11}$$

and

$$\sum_{p|N} \frac{\log(p)}{p} = O(\log\log(N)) \tag{5.12}$$

(see [BL65, p. 20]), we obtain $h_j(N) = O(\log\log(N))$ as $N \to \infty$ for $j = 1, 2$. Using this, we have

$$\begin{aligned} 2g(g-1)e_3(N) &= -2g\left(1 - \frac{6}{N}\right)\left[2\mathscr{C} + 1 + h_1(N) + h_2(N)\right] \\ &= -2g\left(1 - \frac{6}{N}\right) \times O(\log\log(N)) \\ &= O(g \log\log(N)). \end{aligned}$$

In particular, this implies that $2g(g-1)e_3(N) = o(g\log(N))$.

Consequently, by the claims (5.7), (5.8), (5.9), and the facts $e_1(N) = 0$ and $2g(g-1)e_4(N) = o(g\log(N))$, we have

$$\begin{aligned} \mathcal{A}(N) &= -\frac{2g(g-1)}{\varphi(N)} \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^\times} g_{\mathrm{can}}^{\Gamma(N)}(0_v, \infty) \\ &= 2g(g-1)\Big(m(N) + e_1(N) + e_2(N) + e_3(N) + e_4(N)\Big) \\ &= 4g\log(N) + o(g\log(N)). \end{aligned}$$

Finally, let us consider the geometric contribution $\mathcal{G}(N)$ of $\overline{\omega}^2_{\mathcal{X}_{\Gamma(N)}/\mathbb{Z}[\zeta_N]}/\varphi(N)$, namely, we have

$$\mathcal{G}(N) := \frac{2g(V_0, V_\infty)_{\mathrm{fin}} - (V_0, V_0)_{\mathrm{fin}} - (V_\infty, V_\infty)_{\mathrm{fin}}}{2\varphi(N)(g-1)}$$

(see Proposition 3.4.2). From Lemma 5.2.1 part (c), we know that

$$\begin{aligned}
\mathcal{G}(N) &= 4\left(1-\frac{6}{N}\right)\left(g\sum_{p|N}\frac{p\log(p)}{p^2-1}+\sum_{p|N}\frac{p^2\log(p)}{p^2-1}\right)\\
&= 4\left(1-\frac{6}{N}\right)\left(\log(N)+g\sum_{p|N}\frac{p\log(p)}{p^2-1}+\sum_{p|N}\frac{\log(p)}{p^2-1}\right).
\end{aligned}$$

We claim that

$$\mathcal{G}(N) = o(g\log(N))$$

as $N \to \infty$. Indeed, we have

$$\sum_{p|N}\frac{p\log(p)}{p^2-1} = \frac{1}{2}\sum_{p|N}\frac{\log(p)}{p+1}+\frac{1}{2}\sum_{p|N}\frac{\log(p)}{p-1} = O(\log\log(N)),$$

$$\sum_{p|N}\frac{\log(p)}{p^2-1} = \frac{1}{2}\sum_{p|N}\frac{\log(p)}{p-1}-\frac{1}{2}\sum_{p|N}\frac{\log(p)}{p+1} = O(\log\log(N)),$$

which follows from (5.10), (5.11), and (5.12). As a result, we obtain

$$\begin{aligned}
\frac{\mathcal{G}(N)}{g\log(N)} &= \frac{4}{g\log(N)}\left(1-\frac{6}{N}\right)\left(\log(N)+g\sum_{p|N}\frac{p\log(p)}{p^2-1}+\sum_{p|N}\frac{\log(p)}{p^2-1}\right)\\
&= \frac{4}{g\log(N)}\left(1-\frac{6}{N}\right)\Big(\log(N)+O(g\log\log(N))\Big)\\
&= 4\left(1-\frac{6}{N}\right)\left(\frac{1}{g}+O\Big(\frac{\log\log(N)}{\log(N)}\Big)\right).
\end{aligned}$$

Since the right hand side goes to zero as N tends to infinity, the claim follows. This concludes the proof. $\square$

Appendix A

Special functions I

A.1 The Gamma function

Let $f(t)$ be a function with $t > 0$, satisfying $f(t) = O(t^{-\alpha})$ as $t \to \infty$ for all $\alpha \in \mathbb{R}$, and $f(t) = O(t^{-\beta})$ as $t \to 0$ for some $\beta \in \mathbb{R}$. Then the *Mellin transform of f* is defined by

$$\mathcal{M}[f](s) \coloneqq \int_0^\infty f(t)t^{s-1}\mathrm{d}t,$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \beta$. Observe that it defines a holomorphic function in this half-plane.

Definition A.1.1. The *Gamma function* $\Gamma(s)$ is the Mellin transform of the function e^{-t}, i.e., we have

$$\Gamma(s) \coloneqq \int_0^\infty e^{-t}t^{s-1}\mathrm{d}t,$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

The function $\Gamma(s)$ can be meromorphically continued to the whole complex plane, with simple poles at the points $s = 0, -1, -2, \ldots$; namely, we have

$$\Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(s+n)} + \int_1^\infty e^{-t}t^{s-1}\mathrm{d}t$$

(see [Leb72, p. 3]). Furthermore, if we write also $\Gamma(s)$ for the meromorphic continuation, then it satisfies the following relations

$$\Gamma(s+1) = s\,\Gamma(s);$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)};$$

$$2^{2s-1}\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}\,\Gamma(2s).$$

In addition, it can be easily verified that $\Gamma(s)$ has no zeros in the complex plane; therefore, the function $1/\Gamma(s)$ is entire.

A.2 The Riemann zeta function

Definition A.2.1. The *Riemann zeta function* is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Note that on the half-plane $\mathrm{Re}(s) > 1$, $\zeta(s)$ defines a holomorphic function.

Now we proceed to relate $\zeta(s)$ with $\Gamma(s)$. Let us consider $\Gamma(s/2)$. Making the change of variables $t = n^2\pi x$ in the definition of $\Gamma(s/2)$, then we have

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} \mathrm{d}x,$$

and summing over all $n \in \mathbb{N}$, we obtain

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} \Big(\sum_{n=1}^{\infty} e^{-n^2\pi x} \Big) \mathrm{d}x.$$

This identity provides the meromorphic continuation of $\zeta(s)$ to the whole complex plane, namely, we have

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}} \right) \Big(\sum_{n=1}^{\infty} e^{-n^2\pi x} - 1 \Big) \mathrm{d}x$$

(see [Dav80, pp. 61–62]).

The Riemann zeta function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, i.e., at the point $s = 1$ we have

$$\zeta(s) = \frac{1}{s-1} + \gamma_{\mathrm{EM}} + O(s-1),$$

where γ_{EM} denotes the *Euler–Mascheroni constant*:

$$\gamma_{\mathrm{EM}} := \lim_{n\to\infty} \Big(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n) \Big).$$

We end this part with two useful Laurent expansions at $s = 1$ of functions involving $\Gamma(s)$ and $\zeta(s)$. If $\mathscr{C}$ is given by

$$\mathscr{C} \coloneqq 1 - \log(4\pi) + \frac{\zeta'(-1)}{\zeta(-1)},$$

then we have

$$\zeta(2s-1) = \frac{1/2}{s-1} + \gamma_{\mathrm{EM}} + O(s-1), \tag{A.1}$$

$$\sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)\zeta(2s)} = \frac{6}{\pi} + \frac{12}{\pi}\Big(\mathscr{C} - \gamma_{\mathrm{EM}}\Big)(s-1) + O\Big((s-1)^2\Big). \tag{A.2}$$

A.3 The Legendre functions

The *hypergeometric series* is defined by

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_k} z^n \qquad (|z|<1),$$

where $(\lambda)_0 = 1$ and $(\lambda)_n = n(n+1)\cdots(\lambda+n-1)$.

The Legendre's equation is given by

$$(1-z^2)f'' - 2zf' + \nu(\nu+1)f = 0$$

where ν is an arbitrary real or complex number. There are two particular solutions of this equation (see [Leb72, p. 165]), namely, we have

$$P_\nu(z) \coloneqq F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right) \qquad (|z-1|<2),$$

$$Q_\nu(z) \coloneqq \frac{\sqrt{\pi}\,\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)(2z)^{\nu+1}} F\left(\frac{\nu}{2}+1, \frac{\nu}{2}+\frac{1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

$$(|z|>1, |\arg z|<\pi, \nu \neq -1, -2, \ldots),$$

where $F(\alpha,\beta;\gamma;z)$ is the hypergeometric series. The solutions $P_\nu(z)$ and $Q_\nu(z)$ are called *Legendre functions of the first and second kind*, respectively. Furthermore, the function $P_\nu(z)$ and $Q_\nu(z)$ possess analytic continuations (see [Leb72, (7.3.12), p. 166] and [Leb72, (7.3.20), p. 169]).

The following integral representations of $P_\nu(z)$ and $Q_\nu(z)$ are useful

$$P_\nu(z) = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}t}{(z+\sqrt{z^2-1}\cos(t))^{\nu+1}},$$

$$Q_\nu(z) = \int_0^\infty (z + \sqrt{z^2-1}\cosh(t))^{-\nu-1}\mathrm{d}t$$

(see [Leb72, (7.4.2), p. 172] and [Leb72, (7.4.9), p. 174]).

A.4 The Whittaker's function

The *Kummer's function* is defined by

$$M(a,b,z) := \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!},$$

where $a, b \in \mathbb{N}$, $z \in \mathbb{C}$, and $(\lambda)_n$ is given at the beginning of Section A.3. Set

$$M_{\kappa,\mu}(z) := e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu} M\left(\frac{1}{2}+\mu-\kappa, 1+2\mu, z\right).$$

Definition A.4.1. The *Whittaker's function* is defined by

$$W_{\kappa,\mu}(z) := \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)} M_{\kappa,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} M_{\kappa,-\mu}(z),$$

where $-\pi < \arg(z) < \pi$, $\kappa = b/2 - a$, and $\mu = b/2 - 1/2$ (see, e.g., [AS92, 13.1.34, p. 505] or [MOS66, Chapter VII, p. 295]).

A.5 Some useful Laurent expansions

Let N be a square-free positive integer and $s \in \mathbb{C}$. Then we have

$$N^{-s} = \frac{1}{N} - \frac{\log(N)}{N}(s-1) + O\Big((s-1)^2\Big); \tag{A.3}$$

$$\begin{aligned} &N^{1-4s}\prod_{p|N}\frac{1}{1-p^{-2s}} \\ &= \frac{1}{N}\Bigg(\prod_{p|N}\frac{1}{p^2-1}\Bigg)\Bigg(1-2\Bigg(\sum_{p|N}\frac{p^2\log(p)}{p^2-1}+\log(N)\Bigg)(s-1)+O\Big((s-1)^2\Big)\Bigg); \end{aligned} \tag{A.4}$$

$$\begin{aligned} &\prod_{p|N}\frac{p^s-p^{1-s}}{p^{2s}-1} \\ &= \Bigg(\prod_{p|N}\frac{1}{p+1}\Bigg)\Bigg(1+\Bigg(\sum_{p|N}\frac{1+2p-p^2}{p^2-1}\log(p)\Bigg)(s-1)+O\Big((s-1)^2\Big)\Bigg). \end{aligned} \tag{A.5}$$

Appendix B

Special functions II

B.1 The Selberg/Harish–Chandra transform

Let $\phi \in \mathcal{C}^4(\mathbb{R}_{>0})$ such that there exist real constants $C > 0$ and $\alpha > 1$ satisfying

$$|\phi^{(j)}(t)| \leq \frac{C}{(t+1)^{\alpha+j}},$$

for all $t \in \mathbb{R}_{>0}$ and $j = 0, 1, 2, 3, 4$. For what follows, let $k = 0$ or 2.

Definition B.1.1. The *Selberg/Harish–Chandra transform of weight k of ϕ* is defined by

$$h(r) := \int\limits_{-\infty}^{\infty} g(u) e^{iru} \mathrm{d}r,$$

where

$$\begin{aligned} g(u) &= Q(e^u + e^{-u} - 2), \\ Q(w) &= \int\limits_{-\infty}^{\infty} \phi(w + v^2) \left[\frac{\sqrt{w+4} - iv}{\sqrt{w+4} + iv} \right]^{k/2} \mathrm{d}v \qquad (w \geq 0). \end{aligned}$$

Now let us define the inverse of the Selberg/Harish–Chandra transform. Suppose that $h : \mathbb{R} \longrightarrow \mathbb{C}$ is a function satisfying the following conditions

(i) $h(r) = h(-r)$, for $r \in \mathbb{R}$;

(ii) h admits a holomorphic continuation to the strip $|\mathrm{Im}(r)| < A/2$, for some real $A > 1$;

(iii) h is of rapid decay in this strip.

Definition B.1.2. The *Selberg/Harish–Chandra transform of weight k of h* is defined by

$$\phi_k(t) \coloneqq -\frac{1}{\pi}\int\limits_{-\infty}^{\infty} Q'(t+x^2)\left[\frac{\sqrt{t+4+x^2}-x}{\sqrt{t+4+x^2}+x}\right]^{k/2}\mathrm{d}x \qquad (t\geq 0),$$

where

$$Q(e^u+e^{-u}-2)=g(u),$$

$$g(u)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} h(r)e^{-iru}\mathrm{d}r.$$

B.2 The function $I_k(s;l)$

In the sequel, we let $l\in\mathbb{Z}$, $k=0$ or 2, and set $h(r)$ as follows

$$h(r)\coloneqq e^{-t\left(\frac{1}{4}+r^2\right)}$$

with $t>0$ a fixed real number. This function has a holomorphic continuation to the strip $|\mathrm{Im}(r)|<A/2$, for some $A>1$ real. Denote by ϕ_k the inverse Selberg/Harish–Chandra transform of weight k of h. Let $\pi_k(z,w)$ be the point-pair invariant of weight k given by

$$\pi_k(z,w)\coloneqq\left(\frac{w-\bar{z}}{z-\bar{w}}\right)^{k/2}\phi_k\left(\frac{|z-w|^2}{4\,\mathrm{Im}(z)\mathrm{Im}(w)}\right),$$

and set

$$\nu_k(\gamma;z)\coloneqq\left(\frac{cz+d}{|cz+d|}\right)^k\pi_k(z,\gamma z),$$

where $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{R})$.

Let us consider the function

$$I_k(s;2)\coloneqq\int\limits_{\mathbb{H}}\left[\nu_k\Big(\left(\begin{smallmatrix} 1 & 0\\ 1 & 1\end{smallmatrix}\right);z\Big)+\nu_k\Big(\left(\begin{smallmatrix} -1 & 0\\ 1 & -1\end{smallmatrix}\right);z\Big)\right]\mathrm{Im}(z)^s\mu_{\mathrm{hyp}}(z). \tag{B.1}$$

Lemma B.2.1. *Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)<A$. Then the following identity holds*

$$\frac{\Gamma(s)}{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}\Big(I_2(s;2)-I_0(s;2)\Big)=A_2(t)(s-1)+C_1(t)(s-1)^2+O\Big((s-1)^3\Big),$$

where

$$A_2(t)=-\frac{1}{2}h\Big(\frac{i}{2}\Big)+\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\frac{h(r)}{\frac{1}{4}+r^2}\mathrm{d}r \tag{B.2}$$

and $C_1(t)$ depends only on t and has finite limit C_1 as $t \to \infty$.

Proof. For the proof, we refer the reader to [AU97, Proposition 3.3.4, p. 55]. $\square$

Similarly, suppose that $l \in \mathbb{Z}$ is such that $l > 2$ and define

$$I_k(s;l) := \int\limits_{\mathbb{H}} \left[\nu_k\Big(\left(\begin{smallmatrix} l/2 & (l/2)^2-1 \\ 1 & l/2 \end{smallmatrix}\right); z\Big) + \nu_k\Big(\left(\begin{smallmatrix} -l/2 & (l/2)^2-1 \\ 1 & -l/2 \end{smallmatrix}\right); z\Big)\right] \mathrm{Im}(z)^s \mu_{\mathrm{hyp}}(z). \tag{B.3}$$

Lemma B.2.2. *Let $l \in \mathbb{Z}$ be a given integer such that $l > 2$ and suppose that $s \in \mathbb{C}$ with $\mathrm{Re}(s) < A$. Then the following identity holds*

$$I_2(s;l) - I_0(s;l) = A_l(t)(s-1) + O\Big((s-1)^2\Big),$$

where

$$A_l(t) = -\frac{\pi}{2\eta_l} h\Big(\frac{i}{2}\Big) + \frac{1}{4} \int\limits_{-\infty}^{\infty} \frac{h(r)}{\frac{1}{4}+r^2} e^{-2ir\log(\eta_l)} \mathrm{d}r \tag{B.4}$$

and $\eta_l = (l + \sqrt{l^2-4})/2$.

Proof. For the proof, we refer the reader to [AU97, Proposition 3.3.2, p. 51]. $\square$

Lemma B.2.3. *Let $l \in \mathbb{Z}$ be a given integer such that $l > 2$. Suppose that $h(r) = e^{-t\left(\frac{1}{4}+r^2\right)}$ and $g_u(v) = (1/\sqrt{4\pi u}) e^{-\frac{u}{4}-\frac{v^2}{4u}}$. Then the following identity holds*

$$A_l(t) = -\frac{\pi}{2} \int\limits_0^t g_u(2\log(\eta_l)) \mathrm{d}u.$$

B.3 The function $\Psi_k(y)$

With the assumptions and notations of the previous section, we define for $k = 0$ or 2 and a real variable y the function

$$\Psi_k(y) := \int\limits_{-\infty}^{\infty} \phi_k(u^2) \Big(\frac{2-iu}{2+iu}\Big)^{k/2} e^{2i\pi uy} \mathrm{d}u,$$

Let us set

$$\Psi_k^{\pm}(y) := \Psi_k(y) + \Psi_k(-y). \tag{B.5}$$

Lemma B.3.1. *Let* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 0$*. Then the following identity holds*

$$\int_0^\infty \Psi_0^\pm(y) y^{s-1} \mathrm{d}y = -i \frac{\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})}{2^{s+1}\pi^{s+\frac{3}{2}}} \int_{-\infty}^\infty \frac{\Gamma(\frac{s}{2} - ir)}{\Gamma(1 - \frac{s}{2} - ir)} r h(r) \mathrm{d}r.$$

Proof. For the proof, we refer the reader to [Zag81a, p. 329]. □

Lemma B.3.2. *Let* $s \in \mathbb{C}$ *with* $1 < \mathrm{Re}(s) < \inf\{A, 3/2\}$*. Then the following identity holds*

$$\int_0^\infty \Psi_2^\pm(y) y^{s-1} \mathrm{d}y = -i \frac{\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})}{2^{s+1}\pi^{s+\frac{3}{2}}} \int_{-\infty}^\infty \frac{\Gamma(\frac{s}{2} - ir)}{\Gamma(1 - \frac{s}{2} - ir)} r h(r) \mathrm{d}r + \frac{4\Gamma(s+1)}{(4\pi)^{s+1}} g(0)$$

$$+ \frac{\Gamma(s+1)}{(4\pi)^s} h\left(\frac{i}{2}\right) - \frac{2\Gamma(s+1)}{(4\pi)^{s+1}} \int_{-\infty}^\infty \frac{h(r)}{\frac{1}{4} + r^2} \mathrm{d}r - \frac{\Gamma(s)\sin(\frac{s\pi}{2})}{2^{2s-3}\pi^{s+1}} g(0) \int_0^\infty \frac{\sinh(s\theta)}{\cosh^2(\theta)} \mathrm{d}\theta$$

$$- \frac{\Gamma(s)\sin(\frac{s\pi}{2})}{2^{2s-3}\pi^{s+1}} \int_0^\infty g(u) \sinh\left(\frac{u}{2}\right) \int_{-\infty}^\infty \frac{\mathrm{d}\theta}{\cosh(\theta)\Big(\cosh(\theta)\cosh(u/2) + \sinh(\theta)\Big)^{s+1}} \mathrm{d}u.$$

Proof. For the proof, we refer the reader to [AU97, Proposition 3.2.6, p. 43]. □

Appendix C

Partial zeta functions

Let L be a number field of degree $[L : \mathbb{Q}] = r_1 + 2r_2$ where r_1 denotes the number of real embeddings $\sigma : L \hookrightarrow \mathbb{R}$ and r_2 the number of non-conjugate complex embeddings $\sigma : L \hookrightarrow \mathbb{C}$, and suppose that L has discriminant Δ. Denote by $\mathcal{O}_L$ the ring of integers of L.

C.1 The ray class group

An element $\alpha \in L$ is called *totally positive* if for all the real embeddings σ we have $\sigma(\alpha) > 0$. The norm element will be written as $N_{L/\mathbb{Q}}(\alpha)$, or simply $N(\alpha)$ for short. If $\mathfrak{a} \subset \mathcal{O}_L$ is an integral ideal, the norm $\mathcal{N}\mathfrak{a}$ is given by $\mathcal{N}\mathfrak{a} = [\mathcal{O}_L : \mathfrak{a}]$. Since we have $\mathcal{N}(\mathfrak{a}\mathfrak{b}) = \mathcal{N}\mathfrak{a}\mathcal{N}\mathfrak{b}$, then the norm can be extended to fractional ideals.

A *valuation* is a function $v : L \longrightarrow \mathbb{R}$ satisfying the following conditions

(i) $v(x) \geq 0$ with equality if and only if $x = 0$;

(ii) $v(xy) = v(x)v(y)$;

(iii) $v(x + y) \leq v(x) + v(y)$.

If v satisfies the condition $v(x + y) \leq \max\{v(x), v(y)\}$, then we say that v is *non-archimedean*; otherwise we say that v is *archimedean*.

In what follows, we denote by M_L, M_∞, and M_0 the sets of valuations of L, archimedean valuations of L, and non-archimedean valuations of L, respectively.

Definition C.1.1. A *cycle of* L is a formal product of valuations of L

$$\mathfrak{m} = \prod_{v \in M_L} v^{m(v)}$$

where $m(v)$ is a non-negative integer with $m(v) = 0$ for all but a finite number of v. Furthermore, $m(v) \in \{0, 1\}$ if v is real and $m(v) = 0$ if v is complex.

Remark C.1.2. Any cycle $\mathfrak{m}$ of L can be rewritten in the following form

$$\begin{aligned} \mathfrak{m} &= \prod_{v \in M_\infty} v^{m(v)} \times \prod_{v \in M_0} v^{m(v)} \\ &= \mathfrak{m}_\infty \cdot \mathfrak{m}_0 \end{aligned}$$

where $\mathfrak{m}_\infty$ refers to a formal product of real embeddings of L and $\mathfrak{m}_0$ is the formal product over the non-archimedean valuations. Since each valuation in $\mathfrak{m}_0$ comes from a prime ideal $\mathfrak{p} \subset \mathcal{O}_L$, then we can identify $\mathfrak{m}_0$ with the ideal

$$\prod_{v_\mathfrak{p} \in M_0} \mathfrak{p}^{m(v_\mathfrak{p})}.$$

In what follows, we use this identification without mention it.

Notation C.1.3. In the sequel, we write $\mathfrak{p} | \mathfrak{m}_0$ to indicate that the valuation $v_\mathfrak{p}$ has exponent $m(v_\mathfrak{p}) > 0$ and similarly for $v | \mathfrak{m}_\infty$.

Suppose that $\mathfrak{p} \subset \mathcal{O}_L$ is a prime ideal. We will write $\mathcal{O}_\mathfrak{p}$ for the localization of $\mathcal{O}_L$ at $\mathfrak{p}$ and we let $\mathfrak{m}_\mathfrak{p} \subset \mathcal{O}_\mathfrak{p}$ be its maximal ideal.

Definition C.1.4. Let $\mathfrak{m}$ be a cycle of L. An element $\xi \in L$ is *congruent to* 1 mod* $\mathfrak{m}$, denoted by $\xi \equiv 1 \text{ mod}^* \mathfrak{m}$, if it satisfies the following conditions:

(i) If each prime ideal $\mathfrak{p} \subset \mathcal{O}_L$ such that $\mathfrak{p} | \mathfrak{m}_0$, then $\xi \in \mathcal{O}_\mathfrak{p}$ and also $\xi \equiv 1 \bmod \mathfrak{m}_\mathfrak{p}^{m(v_\mathfrak{p})}$,

(ii) If $v | \mathfrak{m}_\infty$ is an archimedean valuation and σ_v is the corresponding embedding of L in $\mathbb{R}$, then

$$\sigma_v(\xi) > 0.$$

Let $\mathfrak{m}$ be a cycle of L. Denote by $I(\mathfrak{m})$ the group of fractional ideals of L relatively prime to $\mathfrak{m}_0$ and by $P_\mathfrak{m}$ the subgroup of principal ideals (ξ) with $\xi \equiv 1 \text{ mod}^* \mathfrak{m}$, respectively.

Definition C.1.5. Let $\mathfrak{m}$ be a cycle of L. The *ray class group* $Cl_\mathfrak{m}(L)$ of L is defined by

$$Cl_\mathfrak{m}(L) := I(\mathfrak{m})/P_\mathfrak{m}.$$

An element of $Cl_\mathfrak{m}(L)$ is called $\mathfrak{m}$-ideal class.

Remark C.1.6. If we take $\mathfrak{m}_\infty$ as the empty product and $\mathfrak{m}_0 = (1)$, we see that $I(\mathfrak{m})$ is the group I of all fractional ideals of L, and $P_\mathfrak{m}$ degenerates to the subgroup P of all principal ideals. Therefore, $Cl_{\mathfrak{m}}(L) = Cl(L)$, the classical class group. Likewise, let $\mathfrak{m}_\infty$ be the product of all real embedding of the number field L and $\mathfrak{m}_0 = (1)$. Then, $I(\mathfrak{m})$ is equal to I, but $P_{\mathfrak{m}}$ is now the subgroup of principal ideals generated by totally positive elements. In this case, we have $Cl_{\mathfrak{m}}(L) = Cl_+(L)$, the narrow class group of L.

C.2 Partial zeta functions

Definition C.2.1. Let $\mathfrak{C} \in Cl_{\mathfrak{m}}(L)$ be an $\mathfrak{m}$-ideal class. The *partial zeta function associated to* $\mathfrak{C}$ is given by

$$\zeta(s, \mathfrak{C}) := \sum_{\mathfrak{a} \in \mathfrak{C}} \frac{1}{(\mathcal{N}\mathfrak{a})^s},$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and the sum runs over all nonzero integral ideals $\mathfrak{a} \subset \mathcal{O}_L$ contained in the $\mathfrak{m}$-ideal class $\mathfrak{C}$.

Suppose that $\mathfrak{m}$ is a cycle of L containing $s(\mathfrak{m})$ real embeddings satisfying $m(v) > 0$. Then we define its norm as follows

$$\mathcal{N}(\mathfrak{m}) := 2^{s(\mathfrak{m})} \cdot \mathcal{N}(\mathfrak{m}_0).$$

Let $U_{\mathfrak{m}}$ be the subgroup of $\mathcal{O}_L^\times$ which are congruent to 1 mod* $\mathfrak{m}$. It can be proved that $U_{\mathfrak{m}}$ is generated by $l := r_1 + r_2 - 1$ elements, say $\varepsilon_{\mathfrak{m}}^{(i)}$, $i = 1, \ldots, l$. If $\{\sigma_1, \ldots, \sigma_{r_1}\}$ resp. $\{\sigma_{r_1+1}, \ldots \sigma_{r_1+r_2}\}$ is set of the real and non-conjugate complex embeddings of L respectively, then the $\mathfrak{m}-$*regulator* $R_{\mathfrak{m}}$ is defined by

$$R_{\mathfrak{m}} := |\det(l_i(\varepsilon_{\mathfrak{m}}^{(j)}))|,$$

where $l_i(\xi)$ are the coordinates of the map

$$l(\xi) := (\log(|\sigma_1(\xi)|), \ldots, \log(|\sigma_{r_1}(\xi)|), \log(|\sigma_{r_1+1}(\xi)|^2), \ldots, \log(|\sigma_{r_1+r_2}(\xi)|^2)).$$

Theorem C.2.2. *Let L be a number field of degree n and $\mathfrak{C} \in Cl_{\mathfrak{m}}(L)$ be an $\mathfrak{m}$-ideal class. Suppose that $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 1 - 1/n$. *Then the partial zeta function $\zeta(s, \mathfrak{C})$ defines a meromorphic function on* $\mathrm{Re}(s) > 1 - 1/n$ *with a unique simple pole at $s = 1$ with residue*

$$\lim_{s \to 1} (s-1)\zeta(s, \mathfrak{C}) = \frac{2^{r_1}(2\pi)^{r_2} R_{\mathfrak{m}}}{\omega_{\mathfrak{m}} \cdot \mathcal{N}(\mathfrak{m}) |\Delta|^{1/2}},$$

where $\omega_{\mathfrak{m}}$ is the number of roots of unity in $U_{\mathfrak{m}}$.

Proof. For the proof we refer the reader to [Lan94, Theorem 5c, p. 161]. □

Remark C.2.3. In particular, if L is a totally real quadratic field, and the cycle $\mathfrak{m}$ has infinity part $\mathfrak{m}_\infty$ equal to the product of all the real embeddings of L, then $s(\mathfrak{m}) = 2$, $r_1 = 2$, $r_2 = 0$, and $U_\mathfrak{m}$ is generated by just one element $\varepsilon_\mathfrak{m}$. Therefore, the formula above simplifies to

$$\lim_{s\to 1}(s-1)\zeta(s,\mathfrak{C}) = \frac{\log(\varepsilon_\mathfrak{m})}{\mathcal{N}(\mathfrak{m}_0)|\Delta|^{1/2}}.$$

Appendix D

L-functions

D.1 Dirichlet characters

Definition D.1.1. Let $N \geq 1$ be an integer. A *Dirichlet character modulo N* is a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$.

Given a Dirichlet character χ modulo N, note that we can extend the domain of definition to $\mathbb{Z}$ writing

$$\chi(n) := \begin{cases} \chi(n \bmod N), & \gcd(n, N) = 1; \\ 0, & \gcd(n, N) > 1. \end{cases}$$

The *principal character modulo N* or simply the *principal character*, denoted by χ_0, is defined as follows

$$\chi_0(n) := \begin{cases} 1, & \gcd(n, N) = 1; \\ 0, & \gcd(n, N) > 1. \end{cases}$$

We say that a Dirichlet character χ is *even* if $\chi(-1) = 1$. In contrast, we say that χ is *odd* if we have $\chi(-1) = -1$.

Note that

$$\overline{\chi}(n) = \chi(n)^{-1} = \chi(n^{-1}), \tag{D.1}$$

where n^{-1} means the inverse modulo N. Finally, the following *orthogonality relations* hold

$$\frac{1}{\varphi(N)} \sum_{n \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(n) = \begin{cases} 1, & \chi = \chi_0; \\ 0, & \chi \neq \chi_0; \end{cases}$$

$$\frac{1}{\varphi(N)} \sum_{\chi \bmod N} \chi(n)\overline{\chi}(a) = \begin{cases} 1, & n \equiv a \bmod N; \\ 0, & n \not\equiv a \bmod N. \end{cases}$$

D.2 Dirichlet L-functions

Definition D.2.1. Let N be a positive integer. A *Dirichlet L-function* is defined by

$$L(s,\chi) \coloneqq \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character modulo N and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Let $L(s,\chi)$ be a Dirichlet L-function. On the half-plane $\mathrm{Re}(s) > 1$, we can write $L(s,\chi)$ as an *Euler product*:

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product runs over all prime numbers. In particular, if we consider the principal character, then we have the following identity

$$L(s,\chi_0) = \zeta(s) \prod_{p|N} \left(1 - \frac{1}{p^s}\right),$$

where $\zeta(s)$ denotes the Riemann zeta function. Observe that this identity also provides the meromorphic continuation of $L(s,\chi_0)$ to the complex plane. In contrast, if $\chi \neq \chi_0$, it can be proved that $L(s,\chi)$ defines an entire function. Furthermore, we have

$$L(1,\chi) \neq 0$$

(see, e.g., [Zag81b, Satz, p. 42]).

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 20. Oktober 2016 Miguel Daygoro Grados Fukuda